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On the Computation of One-Loop Amplitudes with External Fermions in 4D Heterotic Superstrings

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ABSTRACT

We present the technical tools needed to compute any one-loop amplitude involving external spacetime fermions in a four-dimensional heterotic string model à la Kawai-Lewellen-Tye. As an example, we compute the one-loop three-point amplitude with one "photon" and two external massive fermions ("electrons"). As a check of our computation, we verify that the one-loop contribution to the Anomalous Magnetic Moment vanishes if the model has spacetime supersymmetry, as required by the supersymmetric sum rules.

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Introduction and Summary

String theory [1], since its beginning more than twenty years ago, has been a very interesting arena for the development of new ideas in theoretical high-energy physics and appears to be the only candidate for the unification of all elementary interactions. This being the case, it could then seem strange that rather few computations of one-loop amplitudes have ever been performed in string models. There are various reasons for this. One is that string theories are naturally formulated at the string scale which is of the order of magnitude of the Planck scale. Thus, the interesting phenomenology at experimentally accessible energy scales is described by a low-energy effective field theory for the string's massless modes. Another reason is that computations of string amplitudes turn out to be very long and tricky. Indeed, as of today, full one-loop amplitudes have been explicitly computed only for external spacetime bosons.

The purpose of this paper is to address the (mostly technical) problem of computing loop amplitudes with external space-time fermions in a four-dimensional heterotic string theory. Such computations, although not of a direct phenomenological importance, can have several interesting applications.

Obviously, the computation of string loop amplitudes will give a better understanding of the properties and characteristics of string theories per se. Among the various interesting issues is the full understanding of the analytical properties of string amplitudes [2,3,4], their divergencies and the associated renormalization [5,6,7]. Clearly a full discussion of these points would require an off-shell formulation, or at least computation, of string amplitudes.

Another point that can have direct consequences for phenomenology and in general for our understanding of field theory, is that in the low energy ($\alpha' \to 0$) limit, a string theory becomes a bona fide field theory. This limit is not in any way straightforward, since, for example, in closed string theories at each loop order any amplitude is given by only one "diagram".

Interesting are, for example, the results obtained by Kaplunovsky [8] who computed

the one-loop Yang-Mills beta function and the threshold corrections from string theory. Moreover, Bern and Kosower [9] were able to obtain new (simplified) Feynman-like rules for one-loop computations in pure Yang-Mills theory. It would obviously be interesting to extend their results to the full QCD theory.

The necessary tools for computing loop amplitudes involving external spacetime fermions are known in principle [10], but the technical difficulties involved, particularly in four dimensional heterotic models, are quite considerable. We choose to work with string models constructed using free world-sheet fermions à la Kawai-Lewellen-Tye (KLT) [11] (see also [12,13]). The basic problem is then to compute free-fermion correlation functions on an arbitrary Riemann surface in the presence of spin field operators. The well known way of doing this is by bosonization [14,15] (but for a different approach see ref. [16]). But even in this case the computation of an amplitude remains a non trivial task and in this paper we present the technical tools needed to compute any amplitude involving external spacetime fermions. We restrict ourselves to one-loop amplitudes but would like to stress that the generalization to multiloop amplitudes is straightforward.

The main technical point concerns the bosonization itself: In bosonizing the world-sheet fermions one needs to introduce cocycles to guarantee the correct anti-commutation relations. The cocycles play a fundamental role in reconstructing the Lorentz algebra and the explicit Lorentz covariance of the final result, which is lost when the amplitude is written in bosonized form. However, as was discussed already in ref. [15], cocycles are in general not uniquely defined but can be introduced in many different ways, not all of which are physically acceptable. We determine which are the conditions that a proper set of cocycles must satisfy and present an explicit solution in the context of a particular (KLT) *toy* model. Only given such a solution can the bosonization procedure be said to be completely well-defined.

Anyway, this is not yet enough to reconstruct the Lorentz covariance of the final result. Indeed it turns out that correlators involving for example the Lorentz Kač-Moody current $\psi^\mu\psi^\nu(z)$ involve different expressions in terms of theta functions, depending on the values of the Lorentz indices μ, ν. Then, to achieve explicit Lorentz covariance, one needs to prove some non-trivial identities in theta functions.

We also consider the correct normalization of the string amplitudes. We offer a general formula for the N-string one-loop amplitude with the correct overall normalization. One still has to properly normalize the vertex operators. This may be done, case by case, using

the method advocated in ref. [17]. An example is provided in appendix D.

For a generic amplitude, the final form at which we would arrive still leaves to do the sum over the spin structures and the integral over the moduli. This resembles the stage in a field theory calculation where loop momentum integrals and internal Lorentz algebra has been performed, leaving only an integral over Schwinger proper times (or Feynman parameters) — except, of course, that in field theory we have the contribution of many diagrams. In general neither the summation over the spin structures nor the integral over the moduli can be done analytically, but in simple cases it is possible to evaluate them numerically.

As a non-trivial check of the correctness and consistency of our approach, and in order to present the reader with a relatively *simple* example, we explicitly compute a one-loop three-point amplitude in our KLT toy model, involving one $U(1)$ gauge boson (a "photon") and two "electrons", that is, spin $\frac{1}{2}$ particles with mass of the order of the Planck mass and nonzero $U(1)$ charge. One of the terms in this amplitude gives the one-loop contribution to the Anomalous Magnetic Moment (AMM) of the "electron". Recently, Ferrara and Porrati [18] have proven some Supersymmetric Sum Rules which state that in a model with $N = 1$ space-time supersymmetry the anomalous magnetic moment for particles of spin $\frac{1}{2}$ in $(0, \frac{1}{2})$ multiplets, vanishes. In other words, the tree-level value for the gyromagnetic ratio, $g = 2$, does not receive any corrections. The toy model we have chosen to work with has the particular property that the spectrum is either $N = 1$ supersymmetric or non-supersymmetric, depending on the values of certain parameters defining the GSO projections. Checking the Ferrara-Porrati Sum Rules will then provide us with a quite non-trivial check on our computations. (To this end a crucial role is played by a spin-structure dependent phase appearing in the superghost part of the amplitude, as explicitly computed in appendix B. As far as we know this phase has been accounted for only in ref. [19].)

It should be mentioned that, when decomposed in Lorentz structures, the three-point amplitude has two more terms. The first term has the same structure as the tree-level amplitude, and gives rise to various renormalizations [5,6,7]. Since the integral over the moduli diverges, a proper treatment should be done in the context of an off-shell computation and we do not consider it in the present work. The last term has the Lorentz structure of an Electric Dipole Moment, but it turns out not to depend on the sign of the electric charge of the "electron/positron". For this reason we call it a *Pseudo Electric*

Dipole Moment (PEDM): On top of violating P and T, like an ordinary electric dipole moment, this PEDM also violates C. Then it violates CPT. But this should not be possible since it was claimed in ref. [20] (see also [21,22]) that KLT string models do not violate CPT perturbatively. Indeed we will show that for this term in the amplitude the integral over the moduli vanishes, leaving no contradictions. This gives us another, unexpected, non-trivial check on our computations.

The paper is organized as follows. In the first section we review the KLT-formalism for constructing four-dimensional heterotic string models with free world-sheet fermions. Our notations differ somewhat from those of ref. [11] and furthermore, we use the Lorentz covariant (rather than the light-cone) formulation. We then introduce our toy model and discuss its spectrum, the GSO projection conditions and the spacetime supersymmetry.

In the second section we introduce the tools necessary for the computation of arbitrary loop amplitudes involving external spacetime fermions. Thus, we introduce the spin field vertex operators through bosonization of the world-sheet fermions and we discuss in details how to make a consistent and convenient choice of cocycles. In the context of the toy model we introduce the "electron/positron" (and "photon") vertex operators, discuss the related Dirac equation and introduce a generalized charge conjugation matrix.

In the third section we compute the specific one-loop three-point amplitude of two "electrons" and a "photon". We consider the role of the Picture Changing Operators (PCOs) and outline the various steps involved in the computation: The evaluation of the various (world-sheet) correlators, the appearance of the identities in theta functions needed for obtaining a Lorentz covariant result, and the use of the GSO projection conditions and of the Dirac equation. Finally, as the first check of consistency, we show that the amplitude thus obtained does not depend on the point of insertion of the PCO.

In the last section we discuss the vanishing of the Pseudo Electric Dipole Moment and we show how in models with spacetime supersymmetry, the Anomalous Magnetic Moment vanishes, in agreement with the Ferrara-Porrati sum rules.

The appendices contain: A summary of notations, conventions and useful formulæ; some details on the computation of ghost and superghost correlators; a discussion of the Lorentz covariant formulation of the KLT formalism; the computation of the normalization of the "electron/positron" vertex operators; and the proof of one of the identities in theta functions required for the explicit Lorentz covariance.

1. The KLT 4d Heterotic String Models

We start out by briefly reviewing the KLT construction [11] of 4d heterotic string models; our notations differ somewhat from those of ref. [11]. Also, we choose to work in the Lorentz-covariant formulation, rather than the light-cone gauge. We choose Euclidean signature on the space-time metric throughout, only rotating to Minkowski space at the very end of calculations.

1.1 The KLT formalism

A 4-dimensional heterotic KLT model is described in the Lorentz-covariant formulation by the four space-time coordinate fields $X^{\mu}(z,\bar{z})$; twenty-two left-moving complex fermions $\bar{\psi}_{(\bar{l})}(\bar{z})$, $\bar{l}=\bar{1},\ldots,\overline{22}$; eleven right-moving complex fermions $\psi_{(l)}(z)$, $l=0,1;2,\ldots,10$; right-moving superghosts β,γ; and left- and right-moving reparametrization ghosts $\bar{b},\bar{c}$ and b,c.

Corresponding to each of the right-moving complex fermions we define two real fermions by

$$\psi^m_{(l)}=\left\{\tfrac{1}{\sqrt{2}}(\psi_{(l)}+\psi^*_{(l)})\ ,\ \tfrac{1}{i\sqrt{2}}(\psi_{(l)}-\psi^*_{(l)})\right\}\ ,\quad m=1,2\ . \tag{1.1}$$

The four real fermions ψ^{μ} that transform as a space-time vector are related to the complex fermions $\psi_{(0)}$ and $\psi_{(1)}$ by $\psi^0\equiv\psi^1_{(0)}$, $\psi^1\equiv\psi^2_{(0)}$, $\psi^2\equiv\psi^1_{(1)}$ and $\psi^3\equiv\psi^2_{(1)}$, while the nine complex fermions $\psi_{(l)}(z)$, $l=2,\ldots,10$ are called *internal*.

The right-movers possess $N=1$ world-sheet supersymmetry, generated by the supercurrent

$$T_F=T_F^{[X,\psi]}-c\partial\beta-\frac{3}{2}(\partial c)\beta+\frac{1}{2}\gamma b\ , \tag{1.2}$$

where the orbital part is given by

$$T_F^{[X,\psi]}=-\tfrac{i}{2}\partial X\cdot\psi-\tfrac{i}{2}\sum_{m=1}^{2}(\psi^m_{(2)}\psi^m_{(3)}\psi^m_{(4)}+\psi^m_{(5)}\psi^m_{(6)}\psi^m_{(7)}+\psi^m_{(8)}\psi^m_{(9)}\psi^m_{(10)})\ . \tag{1.3}$$

Notice that the supercurrent arranges the nine internal fermions into three *triplets*.

Any KLT model is specified by a certain number of basis vectors $\mathbf{W}_i$ defining the set of possible boundary conditions (spin structures) for the fermions, and a set of parameters k_{ij} defining the GSO projection.

Each entry in the vector $\mathbf{W}_i$ is a rational number and corresponds to one of the complex fermions. Since the fermions $\psi_{(0)}$ and $\psi_{(1)}$ (and the superghosts) are forced to carry the same spin structure (otherwise the supercurrent (1.2) would not have well-defined boundary conditions) we include in the vectors $\mathbf{W}_i$ only the fermions $\bar{\psi}_{(\bar{l})}$, $\bar{l} = \bar{1}, \ldots, \overline{22}$ and $\psi_{(l)}$, $l = 1, \ldots, 10$.

On the cylinder, described by a complex coordinate z, the boundary conditions of the fermions are then specified as follows:

$$\begin{aligned} \psi_{(l)}(e^{2\pi i}z) &= e^{2\pi i(\frac{1}{2}-\alpha_l)}\psi_{(l)}(z) \quad , \quad l = 0, 1, \ldots, 10 \\ \bar{\psi}_{(\bar{l})}(e^{-2\pi i}\bar{z}) &= e^{-2\pi i(\frac{1}{2}-\bar{\alpha}_l)}\bar{\psi}_{(\bar{l})}(\bar{z}) \quad , \quad l = 1, \ldots, 22 \; , \end{aligned} \tag{1.4}$$

where $\alpha_0 \equiv \alpha_1$ and $\bar{\alpha}_l$ ($l = 1, \ldots, 22$) and α_l ($l = 1, \ldots, 10$) are the components of the vector

$$\boldsymbol{\alpha} = \sum_{i=0,1,\ldots} m_i \mathbf{W}_i \equiv m\mathbf{W} \; , \tag{1.5}$$

which is parametrized by integers m_i taking values in $\{0, \ldots, M_i - 1\}$, M_i being the smallest integer such that $M_i \mathbf{W}_i$ (i not summed) is a vector of integer numbers. The Ramond (R) and Neveu-Schwarz (NS) boundary conditions correspond to $\alpha_l = 0$ and $\alpha_l = \frac{1}{2}$ respectively.

Notice that in the class of models we consider, formulated in terms of complex fermions, the requirement that the supercurrent (1.3) has well-defined boundary conditions dictates that all the right-moving fermions satisfy either R or NS boundary conditions and furthermore, that

$$\alpha_1 \overset{\text{MOD }1}{=} \sum_{l=2}^{4} \alpha_l \overset{\text{MOD }1}{=} \sum_{l=5}^{7} \alpha_l \overset{\text{MOD }1}{=} \sum_{l=8}^{10} \alpha_l \tag{1.6}$$

for any set of boundary conditions $\boldsymbol{\alpha}$. For the left-moving fermions boundary conditions other than R or NS are possible.

Each set of integers m_i (each vector $\boldsymbol{\alpha}$) defines a *sector* in the spectrum of string states. We have $\prod_i M_i$ such sectors. For example, the set of basis vectors always include the vector [11]

$$\mathbf{W}_0 \;=\; \left((\tfrac{1}{2})^{22} | (\tfrac{1}{2})(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})^3\right) \; , \tag{1.7}$$

which describes the NS boundary condition for all fermions, and the vector $\mathbf{W} = \mathbf{0}$ which describes the R boundary conditions for all fermions.

The string states in the sector specified by $\boldsymbol{\alpha}$ are space-time bosons (fermions) depending on whether the first right-moving component

$$\alpha_1 = \sum_i m_i (\mathbf{W}_i)_{(1)} \equiv \sum_i m_i s_i \tag{1.8}$$

(which specifies the boundary condition for the supercurrent) takes the value 1/2 (0) mod 1, and we will refer to it as a bosonic (fermionic) sector.

In a bosonic (fermionic) sector, the set of all possible string states are constructed from the superghost vacuum with charge $q' = -1$ ($q' = -1/2$) [14], and the set of *allowed* string states is specified by the GSO projections, which in the Lorentz-covariant formulation assume the form

$$\mathbf{W}_i \cdot \mathbf{N}_{[\![\boldsymbol{\alpha}]\!]} - s_i (N^{(0)}_{[\![\alpha_1]\!]} - N^{(\beta\gamma)}_{[\![\alpha_1]\!]}) \overset{\text{MOD } 1}{=} \sum_j k_{ij} m_j + s_i + k_{0i} - \mathbf{W}_i \cdot [\![\boldsymbol{\alpha}]\!] \ , \tag{1.9}$$

as shown in appendix C.

Here the inner-product of two vectors, such as $\mathbf{W}_i \cdot \mathbf{N}$, includes a factor of (-1) for right-moving components. Also, for any real number α we define $[\![\alpha]\!] \equiv \alpha - \Delta$, where $0 \leq [\![\alpha]\!] < 1$ and $\Delta \in \mathbb{Z}$.

$\mathbf{N}_{[\![\boldsymbol{\alpha}]\!]}$ is the vector of fermion number operators in the sector $\boldsymbol{\alpha}$, $N^{(0)}_{[\![\alpha_1]\!]}$ is the number operator for the "longitudinal" complex fermion $\psi_{(0)}$ and $N^{(\beta\gamma)}_{[\![\alpha_1]\!]}$ is the superghost number operator.

If we introduce mode expansions

$$\begin{aligned} \psi_{(l)}(z) &= \sum_{q \in \mathbb{Z}} \psi^{(l)}_{q - [\![\alpha_l]\!]} z^{-q + [\![\alpha_l]\!] - 1/2} \\ \beta(z) &= \sum_{q \in \mathbb{Z}} \beta_{q - [\![\alpha_1]\!]} z^{-q + [\![\alpha_1]\!] - 3/2} \\ \gamma(z) &= \sum_{q \in \mathbb{Z}} \gamma_{q - [\![\alpha_1]\!]} z^{-q + [\![\alpha_1]\!] + 1/2} \ , \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} \{\psi^{(l)}_{q - [\![\alpha_l]\!]}, \psi^{(l')\,*}_{q' + [\![\alpha_l]\!]}\} &= \delta_{q+q'} \delta^{l,l'} \\ [\gamma_{q - [\![\alpha_1]\!]}, \beta_{q' + [\![\alpha_1]\!]}] &= \delta_{q+q'} \ , \end{aligned}$$

we may write the lth fermion number operator as

$$N^{(l)}_{[\![\alpha_l]\!]} = \sum_{q=1}^{\infty} \left[n^{(l)}_{q + [\![\alpha_l]\!] - 1} - n^{(l)*}_{q - [\![\alpha_l]\!]} \right] \ , \tag{1.11}$$

with the fermion and anti-fermion mode occupation numbers defined by

$$n^{(l)}_{q+[\![\alpha_l]\!]-1} = \psi^{(l)}_{-q-[\![\alpha_l]\!]+1}\psi^{(l)*}_{q+[\![\alpha_l]\!]-1} \ , \qquad n^{(l)*}_{q-[\![\alpha_l]\!]} = \psi^{(l)*}_{-q+[\![\alpha_l]\!]}\psi^{(l)}_{q-[\![\alpha_l]\!]} \ ; \tag{1.12}$$

and

$$N^{(\beta\gamma)}_{[\![\alpha_1]\!]} = -\sum_{q=1}^{\infty}\left[\beta_{-q+[\![\alpha_1]\!]}\gamma_{q-[\![\alpha_1]\!]} + \gamma_{-q+1-[\![\alpha_1]\!]}\beta_{q-1+[\![\alpha_1]\!]}\right] \ . \tag{1.13}$$

Notice that $N^{(\beta\gamma)}_{[\![\alpha_1]\!]} = 0$ for states in the superghost vacuum. The GSO projections (1.9) are parametrized by the quantities k_{ij}. As shown in Ref. [11] consistency at the 1-loop level requires the k_{ij} and the $\mathbf{W}_i$ to satisfy the following conditions

$$\begin{aligned} &k_{ij} + k_{ji} \overset{\text{MOD } 1}{=} \mathbf{W}_i \cdot \mathbf{W}_j \\ &M_j k_{ij} \overset{\text{MOD } 1}{=} 0 \\ &k_{ii} + k_{i0} + s_i - \tfrac{1}{2}\mathbf{W}_i \cdot \mathbf{W}_i \overset{\text{MOD } 1}{=} 0 \ . \end{aligned} \tag{1.14}$$

On the torus the spin structure $\left[\begin{smallmatrix}\alpha_l\\ \beta_l\end{smallmatrix}\right]$ of the fermion (l) is parametrized by two sets of integers, m_i and n_i, each taking values in $\{0, \ldots, M_i - 1\}$:

$$\begin{aligned} \boldsymbol{\alpha} &= \sum_{i=0,1,\ldots} m_i \mathbf{W}_i \\ \boldsymbol{\beta} &= \sum_{i=0,1,\ldots} n_i \mathbf{W}_i \ . \end{aligned} \tag{1.15}$$

The m_i specify the sector of states being propagated in the loop. The n_i specify the boundary conditions when going around the time-like direction of the torus. We sum over the spin structures by summing over the $(\prod_i M_i)^2$ possible values of these integers. The summation over the n_i enforces the GSO projection on the states propagating in the loop. Therefore the sum over spin structures may also be viewed as a sum over the full spectrum of GSO projected states circulating in the loop.

The 1-loop partition function of the KLT model can be written as

$$\begin{aligned} \mathcal{Z} \ = \ \sum_{m_i,n_j} C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \int \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\tau)^2}\,(\bar\eta(\bar\tau))^{-24}\prod_{l=1}^{22}\overline{\Theta}\left[\begin{smallmatrix}\bar\alpha_l\\ \bar\beta_l\end{smallmatrix}\right](0|\bar\tau) \ \times \\ (\eta(\tau))^{-12}\prod_{l=1}^{10}\Theta\left[\begin{smallmatrix}\alpha_l\\ \beta_l\end{smallmatrix}\right](0|\tau)\frac{1}{\mathrm{Im}\tau} \ , \end{aligned} \tag{1.16}$$

where the summation coefficients are given by

$$C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = \frac{1}{\prod_i M_i} \exp\left\{-2\pi i \left[\sum_i (n_i + \delta_{i,0})\left(\sum_j k_{ij} m_j + s_i - k_{i0}\right) + \sum_i m_i s_i + \tfrac{1}{2}\right]\right\} . \tag{1.17}$$

These coefficients are chosen so that all states in the GSO-projected spectrum describing space-time bosons (fermions) contribute to the partition function with weight $+1$ (-1). Using the properties (1.14) it is straightforward to check that the partition function (1.16) is modular invariant. Our expression (1.17) for the summation coefficients is somewhat simpler than that given in ref. [11], thanks to certain phases being absorbed into the definition of the Θ function (see appendix A for conventions).

To conclude this subsection, we recall the "mass formula" [11]. We know that only states satisfying the level-matching condition $L_0 = \bar{L}_0$ can propagate, and these have a mass given by

$$\frac{\alpha'}{4} M^2 = \bar{L}_0 - \frac{\alpha'}{4} p^2 = L_0 - \frac{\alpha'}{4} p^2 . \tag{1.18}$$

For states in the (super) ghost vacuum

$$\begin{aligned} L_0 \; = & \; \frac{\alpha'}{4} p^2 + \sum_{l=0}^{10} \left\{ E_{[\![\alpha_l]\!]} + \sum_{q=1}^{\infty} \Big(\left(q + [\![\alpha_l]\!] - 1\right) n^{(l)}_{q+[\![\alpha_l]\!]-1} \right. \\ & \left. + \left(q - [\![\alpha_l]\!]\right) n^{(l)*}_{q-[\![\alpha_l]\!]} \Big) \right\} + \sum_{q=1}^{\infty} q a_{-q} \cdot a_q - 1 + E^{(\beta\gamma)}_{[\![\alpha_1]\!]} , \end{aligned} \tag{1.19}$$

where a^μ_q are the (right-moving) modes of $X^\mu(z, \bar{z})$, $E^{(\beta\gamma)}_{[\![\alpha_1]\!]}$ is the superghost vacuum energy, which equals $+1/2$ ($+3/8$) in a bosonic (fermionic) sector, and the vacuum energy of the l'th complex fermion (relative to the conformal vacuum) is

$$E_{[\![\alpha_l]\!]} = \tfrac{1}{2}\left([\![\alpha_l]\!] - \tfrac{1}{2}\right)^2 . \tag{1.20}$$

The contribution of minus one represents the reparametrization ghost vacuum energy.

The same formula holds for $\bar{L}_0$, without the superghost vacuum energy, and with the left movers substituted for the right movers.

In each sector, the vacuum energies of the left- and right-movers are given by

$$E_{\text{left}} = \sum_{l=1}^{22} E_{[\![\bar{\alpha}_l]\!]} - 1 , \qquad E_{\text{right}} = \sum_{l=0}^{10} E_{[\![\alpha_l]\!]} - 1 + E^{(\beta\gamma)}_{[\![\alpha_1]\!]} . \tag{1.21}$$

If we restrict ourselves to vectors $\mathbf{W}_i$ where all components are either 0 or 1/2, only NS and R boundary conditions are possible for any given fermion. In the first case, the vacuum is the conformal one, $|0\rangle$; in the second, the vacuum is twofold degenerate and if we represent the zero modes in terms of Pauli matrices

$$\psi_0^{(l)} = \tfrac{1}{2}\left(\sigma_1^{(l)} + i\sigma_2^{(l)}\right) \qquad \text{and} \qquad \psi_0^{(l)*} = \tfrac{1}{2}\left(\sigma_1^{(l)} - i\sigma_2^{(l)}\right) \ , \tag{1.22}$$

the vacua can be labelled $|a_l\rangle$, where $a_l = \pm 1/2$ is the eigenvalue of $\frac{1}{2}\sigma_3^{(l)}$. The fermion number operator (1.11) can then be written as

$$N_0^{(l)} = \sum_{q=1}^{\infty}\left[n_q^{(l)} - n_q^{(l)*}\right] + \tfrac{1}{2}(1+\sigma_3^{(l)}) \tag{1.23}$$

— so the zero mode part counts the state $|-\frac{1}{2}\rangle$ with number zero, and $|+\frac{1}{2}\rangle$ with number one.

1.2 Our toy model

The model we choose to work with has been already proposed in ref. [11]. It has two main advantages: The gauge group contains a $U(1)$; and the spectrum can be made $N=1$ spacetime supersymmetric by choosing appropriate values for the quantities k_{ij}. This means that we can study at the same time spacetime supersymmetric and non-supersymmetric models.

The model is specified by the following boundary vectors

$$\begin{aligned}
\mathbf{W}_0 &= \left((\tfrac{1}{2})^{22}|(\tfrac{1}{2})(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})^3\right) \\
\mathbf{W}_1 &= \left((\tfrac{1}{2})^{22}|(0)(0\tfrac{1}{2}\tfrac{1}{2})^3\right) \\
\mathbf{W}_2 &= \left((\tfrac{1}{2})^{14}(0)^8|(0)(0\tfrac{1}{2}\tfrac{1}{2})(\tfrac{1}{2}0\tfrac{1}{2})^2\right) \\
\mathbf{W}_3 &= \left((\tfrac{1}{2})^7(0)^7(\tfrac{1}{2})^3(0)^5|(0)(\tfrac{1}{2}0\tfrac{1}{2})(0\tfrac{1}{2}\tfrac{1}{2})(\tfrac{1}{2}\tfrac{1}{2}0)\right) \\
\mathbf{W}_4 &= \left((0)^7(0)^7(\tfrac{1}{2})^2(0)(0)^5|(0)(0\tfrac{1}{2}\tfrac{1}{2})(\tfrac{1}{2}\tfrac{1}{2}0)(\tfrac{1}{2}\tfrac{1}{2}0)\right) \ .
\end{aligned} \tag{1.24}$$

Since all entries are 0 or 1/2, $M_i = 2$ and

$$m_i,\, n_j \;=\; \{0,1\} \tag{1.25}$$

with $i,j = 0,\ldots,4$. This implies that the on the torus we have a total of $2^5 \times 2^5 = 1024$ spin structures.

We introduce the shorthand notation

$$\boldsymbol{\alpha} = \sum_{i=0}^{4} m_i \mathbf{W}_i \equiv \mathbf{W}_{\text{subscript}} \ , \tag{1.26}$$

where "subscript" is the list of those i for which $m_i = 1$. For example the sector specified by $m_0 = 1$, $m_1 = 1$, $m_2 = 0$, $m_3 = 0$ and $m_4 = 1$ is called $\mathbf{W}_{014} = \mathbf{W}_0 + \mathbf{W}_1 + \mathbf{W}_4$. The only exception is the sector for which all the m_i are zero which we will just denote by $\boldsymbol{\alpha} = \mathbf{0}$.

The consistency conditions (1.14) are satisfied by any set of k_{ij} satisfying the following matrix equation

$$\begin{pmatrix} k_{00} & k_{01} & k_{02} & k_{03} & k_{04} \\ k_{10} & k_{11} & k_{12} & k_{13} & k_{14} \\ k_{20} & k_{21} & k_{22} & k_{23} & k_{24} \\ k_{30} & k_{31} & k_{32} & k_{33} & k_{34} \\ k_{40} & k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix} \overset{\text{MOD } 1}{=} \begin{pmatrix} k_{00} & k_{01} & k_{02} & k_{03} & k_{04} \\ k_{01} & k_{01} & k_{12} & k_{13} & k_{14} \\ k_{02} & k_{12}+\frac{1}{2} & k_{02} & k_{23} & k_{24} \\ k_{03} & k_{13}+\frac{1}{2} & k_{23} & k_{03}+\frac{1}{2} & k_{34} \\ k_{04} & k_{14}+\frac{1}{2} & k_{24} & k_{34}+\frac{1}{2} & k_{04}+\frac{1}{2} \end{pmatrix} , \tag{1.27}$$

where

$$k_{ij} \ = \ \{0, \tfrac{1}{2}\} \ . \tag{1.28}$$

Hence, the independent k_{ij} can be chosen to be k_{00} and k_{ij} with $i < j$.

From a quick glance at the left-moving part of the vectors (1.24), it is obvious that the world-sheet fermions are grouped together according to $\mathbf{W}_4$: For example the first seven left-moving complex fermions always have the same spin structure; from the corresponding 14 real fermions we may build up the Kač-Moody algebra of $SO(14)$. It is therefore no surprise that the gauge group of our model turns out to be

$$SO(14) \otimes SO(14) \otimes SO(4) \otimes U(1) \otimes SO(10) \ , \tag{1.29}$$

where the $U(1)$ is actually realized as an $SO(2)$. To prove it we need to show that the gluons (massless spin 1 states) existing in the physical spectrum do indeed fill out the adjoint representation of the group (1.29).

1.3 The spectrum

To compute the spectrum of our model we first need to know the vacuum energies of all sectors characterized by the $\{m_i\}$ using eq. (1.21). These are summarized in table

sector	E_{left}	E_{right}	$[\![\alpha_1]\!]$	$[\![\bar{\alpha}_{17}]\!]$
$\mathbf{0}$	$7/4$	$3/4$	0	0
$\mathbf{W}_4$	$3/2$	0	0	0
$\mathbf{W}_3$	$1/2$	0	0	$1/2$
$\mathbf{W}_{34}$	$3/4$	$1/4$	0	$1/2$
$\mathbf{W}_2$	0	0	0	0
$\mathbf{W}_{24}$	$-1/4$	$1/4$	0	0
$\mathbf{W}_{23}$	$1/2$	0	0	$1/2$
$\mathbf{W}_{234}$	$3/4$	$1/4$	0	$1/2$
$\mathbf{W}_1$	-1	0	0	$1/2$
$\mathbf{W}_{14}$	$-3/4$	$1/4$	0	$1/2$
$\mathbf{W}_{13}$	$1/4$	$1/4$	0	0
$\mathbf{W}_{134}$	0	0	0	0
$\mathbf{W}_{12}$	$3/4$	$1/4$	0	$1/2$
$\mathbf{W}_{124}$	1	$1/2$	0	$1/2$
$\mathbf{W}_{123}$	$1/4$	$1/4$	0	0
$\mathbf{W}_{1234}$	0	0	0	0
$\mathbf{W}_0$	-1	$-1/2$	$1/2$	$1/2$
$\mathbf{W}_{04}$	$-3/4$	$1/4$	$1/2$	$1/2$
$\mathbf{W}_{03}$	$1/4$	$1/4$	$1/2$	0
$\mathbf{W}_{034}$	0	0	$1/2$	0
$\mathbf{W}_{02}$	$3/4$	$1/4$	$1/2$	$1/2$
$\mathbf{W}_{024}$	1	0	$1/2$	$1/2$
$\mathbf{W}_{023}$	$1/4$	$1/4$	$1/2$	0
$\mathbf{W}_{0234}$	0	0	$1/2$	0
$\mathbf{W}_{01}$	$7/4$	$1/4$	$1/2$	0
$\mathbf{W}_{014}$	$3/2$	0	$1/2$	0
$\mathbf{W}_{013}$	$1/2$	0	$1/2$	$1/2$
$\mathbf{W}_{0134}$	$3/4$	$1/4$	$1/2$	$1/2$
$\mathbf{W}_{012}$	0	0	$1/2$	0
$\mathbf{W}_{0124}$	$-1/4$	$-1/4$	$1/2$	0
$\mathbf{W}_{0123}$	$1/2$	0	$1/2$	$1/2$
$\mathbf{W}_{01234}$	$3/4$	$1/4$	$1/2$	$1/2$

Table 1: Vacuum energies of the 32 sectors.

1 for our toy model. In this table we also list whether the sector is bosonic or fermionic (whether $[\![\alpha_1]\!] = 1/2$ or 0) and the value $[\![\bar{\alpha}_{17}]\!]$ indicating whether the seventeenth left-moving fermion, which carries the $U(1)$ charge, has R or NS boundary conditions. For sectors with $[\![\bar{\alpha}_{17}]\!] = 0$ the vacuum state carries $U(1)$ charge $\pm 1/2$.

From this table it is simple, sector by sector, to construct the excited states by acting on the vacuum with the creation operators. Again we restrict ourselves to states in the

sector	$\alpha' M^2$	Spin	State	
$\mathbf{W}_0$	-2	0	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\|0\rangle_L \otimes \|0\rangle_R$	$\bar l = 1,\ldots,22$
$\mathbf{W}_{0124}$	-1	0	$\|\bar a_{17,18,\ldots,22}\rangle_L \otimes \|a_{3,4}\rangle_R$	
$\mathbf{W}_{012}$	0	0	$\|\bar a_{15,16,17,18,\ldots,22}\rangle_L \otimes \|a_{5,6,8,9}\rangle_R$	
$\mathbf{W}_2$	0	1/2	$\|\bar a_{15,16,17,18,\ldots,22}\rangle_L \otimes \|\alpha, a_{2,6,9}\rangle_R$	
$\mathbf{W}_{034}$	0	0	$\|\bar a_{1,\ldots,7,17}\rangle_L \otimes \|a_{2,3,5,7}\rangle_R$	
$\mathbf{W}_{134}$	0	1/2	$\|\bar a_{1,\ldots,7,17}\rangle_L \otimes \|\alpha, a_{3,7,8}\rangle_R$	
$\mathbf{W}_{0234}$	0	0	$\|\bar a_{8,\ldots,14,17}\rangle_L \otimes \|a_{2,4,8,10}\rangle_R$	
$\mathbf{W}_{1234}$	0	1/2	$\|\bar a_{8,\ldots,14,17}\rangle_L \otimes \|\alpha, a_{4,5,10}\rangle_R$	
$\mathbf{W}_0$	0	0	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\bar\psi^{\overline{n}}_{-1/2,(\bar k)}\|0\rangle_L \otimes \psi^m_{-1/2,(j)}\|0\rangle_R$	$j = 2,\ldots,10$
$\mathbf{W}_1$	0	1/2	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\bar\psi^{\overline{n}}_{-1/2,(\bar k)}\|0\rangle_L \otimes \|\alpha, a_{2,5,8}\rangle_R$	
$\mathbf{W}_0$	0	1	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\bar\psi^{\overline{n}}_{-1/2,(\bar k)}\|0\rangle_L \otimes \psi^\mu_{-1/2}\|0\rangle_R$	
$\mathbf{W}_1$	0	3/2,1/2	$\bar a^\mu_{-1}\|0\rangle_L \otimes \|\alpha, a_{2,5,8}\rangle_R$	
$\mathbf{W}_0$	0	2,0	$\bar a^\mu_{-1}\|0\rangle_L \otimes \psi^\nu_{-1/2}\|0\rangle_R$	
$\mathbf{W}_0$	0	1	$\bar a^\mu_{-1}\|0\rangle_L \otimes \psi^m_{-1/2,(j)}\|0\rangle_R$	$j = 2,\ldots,10$
$\mathbf{W}_{04}$	1	0	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\bar\psi^{\overline{n}}_{-1/2,(\bar k)}\|\bar a_{15,16}\rangle_L \otimes \|a_{3,4,5,6,8,9}\rangle_R$	$\bar l, \bar k \neq 15, 16$
$\mathbf{W}_{14}$	1	1/2	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\bar\psi^{\overline{n}}_{-1/2,(\bar k)}\|\bar a_{15,16}\rangle_L \otimes \|\alpha, a_{2,3,4,6,9}\rangle_R$	$\bar l, \bar k \neq 15, 16$
$\mathbf{W}_{04}$	1	0	$\bar\psi^{\overline{m}}_{-1,(\bar l)}\|\bar a_{15,16}\rangle_L \otimes \|a_{3,4,5,6,8,9}\rangle_R$	$\bar l = 15, 16$
$\mathbf{W}_{14}$	1	1/2	$\bar\psi^{\overline{m}}_{-1,(\bar l)}\|\bar a_{15,16}\rangle_L \otimes \|\alpha, a_{2,3,4,6,9}\rangle_R$	$\bar l = 15, 16$
$\mathbf{W}_{04}$	1	1	$\bar a^\mu_{-1}\|\bar a_{15,16}\rangle_L \otimes \|a_{3,4,5,6,8,9}\rangle_R$	
$\mathbf{W}_{14}$	1	3/2,1/2	$\bar a^\mu_{-1}\|\bar a_{15,16}\rangle_L \otimes \|\alpha, a_{2,3,4,6,9}\rangle_R$	
$\mathbf{W}_{03}$	1	0	$\|\bar a_{1,\ldots,7,15,16,17}\rangle_L \otimes \|a_{2,4,6,7,8,9}\rangle_R$	
$\mathbf{W}_{13}$	1	1/2	$\|\bar a_{1,\ldots,7,15,16,17}\rangle_L \otimes \|\alpha, a_{4,5,6,7,9}\rangle_R$	
$\mathbf{W}_{023}$	1	0	$\|\bar a_{8,\ldots,16,17}\rangle_L \otimes \|a_{2,3,5,6,9,10}\rangle_R$	
$\mathbf{W}_{123}$	1	1/2	$\|\bar a_{8,\ldots,16,17}\rangle_L \otimes \|\alpha, a_{3,6,8,9,10}\rangle_R$	
$\mathbf{W}_{0124}$	1	0	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\|\bar a_{17,18,\ldots,22}\rangle_L \otimes \psi^m_{-1/2,(k)}\|a_{3,4}\rangle_R$ $\bar l = 1,\ldots,16,\ k = 2, 5,\ldots,10$	
$\mathbf{W}_{24}$	1	1/2	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\|\bar a_{17,18,\ldots,22}\rangle_L \otimes \|\alpha, a_{2,3,4,5,8}\rangle_R$ $\bar l = 1,\ldots,16$	
$\mathbf{W}_{0124}$	1	1	$\bar\psi^{\overline{m}}_{-1/2,(\bar l)}\|\bar a_{17,18,\ldots,22}\rangle_L \otimes \psi^\mu_{-1/2}\|a_{3,4}\rangle_R$	

Table 2: Lighter states in the spectrum, before implementing the GSO projection.

(super) ghost vacuum.

In table 2 we list all such states up to mass level $\alpha' M^2 = 1$. Obviously some of these states will be projected out of the spectrum by the GSO projection. We introduced a

shorthand notation for a set of several R vacua, for example

$$\bar{a}_{17,18,\ldots,22} \equiv \bar{a}_{17}, \bar{a}_{18}, \ldots, \bar{a}_{22} \ , \tag{1.30}$$

where all $\bar{a}_l$ and a_l take values $\pm 1/2$, and $\alpha \equiv (a_0, a_1)$ is a space-time spinor index (not to be confused with the spin structure, of course). Indices $\overline{m}, \overline{n}, m$ take values $1, 2$ and unless otherwise stated, indices $\bar{l}, \bar{k}$ take values $1, \ldots, 22$.

The sector $\mathbf{W}_0$ contains the standard charged tachyon, the would-be gauge bosons, the graviton, dilaton and axion, as well as some further spin 0 and spin 1 states. In the sector $\mathbf{W}_1$ we find gauginos and gravitinos. The number of space-time supersymmetries is given by the number of gravitinos that survive the GSO projection.

1.4 The GSO projection conditions

Now we turn our attention to the GSO projections (1.9). We will demonstrate how they reduce the spectrum of our toy model by means of a few significant examples.

Let us first consider what happens to the states in the $\mathbf{W}_0$-sector. In this sector the GSO projections (1.9) are reduced to

$$\tfrac{1}{2}\left[\sum_{l=1}^{22} \bar{N}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]} - \sum_{l=0}^{10} N^{(l)}_{[\![\alpha_l]\!]}\right] \overset{\text{MOD } 1}{=} \tfrac{1}{2} \tag{1.31}$$

$$\tfrac{1}{2}\left[\sum_{l=1}^{22} \bar{N}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]} - \sum_{l=3,4,6,7,9,10} N^{(l)}_{[\![\alpha_l]\!]}\right] \overset{\text{MOD } 1}{=} 0 \tag{1.32}$$

$$\tfrac{1}{2}\left[\sum_{l=1}^{14} \bar{N}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]} - \sum_{l=3,4,5,7,8,10} N^{(l)}_{[\![\alpha_l]\!]}\right] \overset{\text{MOD } 1}{=} 0 \tag{1.33}$$

$$\tfrac{1}{2}\left[\sum_{l=1}^{7} \bar{N}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]} + \sum_{l=15}^{17} \bar{N}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]} - \sum_{l=2,4,6,7,8,9} N^{(l)}_{[\![\alpha_l]\!]}\right] \overset{\text{MOD } 1}{=} 0 \tag{1.34}$$

$$\tfrac{1}{2}\left[\sum_{l=15}^{16} \bar{N}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]} - \sum_{l=3,4,5,6,8,9} N^{(l)}_{[\![\alpha_l]\!]}\right] \overset{\text{MOD } 1}{=} 0 \ . \tag{1.35}$$

The tachyon has no excitations of the right-movers, and only a single excitation of the left-movers, i.e. $\sum_{l=1}^{22} \bar{N}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]} = 1$ and $N^{(l)}_{[\![\alpha_l]\!]} = 0$ for $l = 0, \ldots, 10$. Thus, it fails to satisfy eq. (1.32) and is projected out.

Now consider the would-be gauge bosons. They have $N^{(0)}_{[\![\alpha_1]\!]} + N^{(1)}_{[\![\alpha_1]\!]} = 1$ and $N^{(l)}_{[\![\alpha_l]\!]} = 0$ for $l = 2, \ldots, 10$. Equations (1.32)-(1.35) then force both of the two left-moving excitations to belong to the same group of fermions: Either $\{\overline{1}, \ldots, \overline{7}\}$, $\{\overline{8}, \ldots, \overline{14}\}$, $\{\overline{15}, \overline{16}\}$, $\{\overline{17}\}$ or $\{\overline{18}, \ldots, \overline{22}\}$. Accordingly, the gauge bosons fill out the adjoint representation of the group (1.29). The "extra" massless spin 1 states, where the vector index is carried by the oscillator $\bar{a}^{\mu}_{-1}$, are all projected out. They have $N^{(l)}_{[\![\alpha_l]\!]} = \delta_{j,l}$ for some $j = 2, \ldots, 10$; by eq. (1.32) this j must be either 2, 5 or 8. By eq. (1.33) it can only be 2. But eq. (1.34) rules out even this possibility.

Next we consider the $\mathbf{W}_1$-sector, in order to see how many of the eight gravitinos and gauginos survive the GSO projection. For the gravitinos only zero mode excitations of the fermions $0, 1, 2, 5, 8$ are allowed. Thus the five projection conditions (1.9) are reduced to

$$\begin{aligned} -\tfrac{1}{2}\left[N^{(0)}_0 + N^{(1)}_0 + N^{(2)}_0 + N^{(5)}_0 + N^{(8)}_0\right] &\overset{\text{MOD } 1}{=} k_{00} + k_{01} + \tfrac{1}{2} \\ 0 &\overset{\text{MOD } 1}{=} 0 \\ -\tfrac{1}{2}\left[N^{(5)}_0 + N^{(8)}_0\right] &\overset{\text{MOD } 1}{=} k_{02} + k_{12} \\ -\tfrac{1}{2}\left[N^{(2)}_0 + N^{(8)}_0\right] &\overset{\text{MOD } 1}{=} k_{03} + k_{13} \\ -\tfrac{1}{2}\left[N^{(5)}_0 + N^{(8)}_0\right] &\overset{\text{MOD } 1}{=} k_{04} + k_{14} \end{aligned} \tag{1.36}$$

from which it follows that a single gravitino survives in the physical spectrum if and only if

$$k_{02} + k_{12} \overset{\text{MOD } 1}{=} k_{04} + k_{14} \ . \tag{1.37}$$

This is then the condition for the model to be $N = 1$ supersymmetric. The same analysis applies to the gauginos, leading again to this condition for spacetime supersymmetry.

It is convenient to rewrite the GSO conditions for the Ramond zero modes in term of Pauli matrices. Consider the generic projection condition

$$\tfrac{1}{2}\left(\sum_{l\in\bar{I}} \bar{N}^{(l)}_0 - \sum_{l\in I} N^{(l)}_0\right) \overset{\text{MOD } 1}{=} r \ , \tag{1.38}$$

where $r \in \{0, 1/2\}$ and the left-hand side involves a total of m number operators. Then, since $N^{(l)}_0 = \frac{1}{2}(1+\sigma^{(l)}_3)$ for zero-mode excitations, this projection condition can be rewritten as

$$\bigotimes_{l\in\{\bar{I},I\}} \sigma^{(l)}_3 \ = \ \exp\left\{2\pi i[r + \tfrac{1}{2}]\right\} \qquad \text{for} \qquad m \text{ odd} \tag{1.39}$$

$$\bigotimes_{l\in\{\bar{I},I\}} \sigma_3^{(l)} \;=\; \exp\{2\pi i r\} \qquad\qquad \text{for} \qquad m \text{ even} \;.$$

For example, the projection conditions for the gravitino can be rewritten as

$$\begin{aligned}
\bigotimes_{l=0,1,2,5,8} \sigma_3^{(l)} &= \exp\{2\pi i\,[k_{00}+k_{01}]\} \\
\bigotimes_{l=5,8} \sigma_3^{(l)} &= \exp\{2\pi i\,[k_{02}+k_{12}]\} \\
\bigotimes_{l=2,8} \sigma_3^{(l)} &= \exp\{2\pi i\,[k_{03}+k_{13}]\} \\
\bigotimes_{l=5,8} \sigma_3^{(l)} &= \exp\{2\pi i\,[k_{04}+k_{14}]\} \;.
\end{aligned} \tag{1.40}$$

Finally we list the GSO projection conditions for the $\alpha' M^2 = 1$ spacetime fermions existing in the $\mathbf{W}_{13}$ sector:

$$\begin{aligned}
\Gamma^5 \otimes \sigma_3^{(5)} &= \exp\left\{2\pi i\left[k_{00}+k_{01}+k_{03}+k_{13}+\tfrac{1}{2}\right]\right\} \\
\sigma_3^{(\overline{17})} \otimes \sigma_3^{(4)} &= \exp\left\{2\pi i\left[k_{02}+k_{12}+k_{13}+k_{23}+k_{04}+k_{14}+k_{34}+\tfrac{1}{2}\right]\right\} \\
\Gamma_{SO(14)} \otimes \sigma_3^{(\overline{17})} \otimes \Gamma^5 \otimes \sigma_3^{(7)} &= \exp\left\{2\pi i\left[k_{00}+k_{01}+k_{03}+k_{04}+k_{14}+k_{34}+\tfrac{1}{2}\right]\right\} \\
\Gamma_{SO(4)} \otimes \sigma_3^{(\overline{17})} \otimes \Gamma^5 \otimes \sigma_3^{(6)} \otimes \sigma_3^{(9)} &= \exp\left\{2\pi i\left[k_{00}+k_{01}+k_{02}+k_{03}+k_{12}+k_{23}+\tfrac{1}{2}\right]\right\},
\end{aligned} \tag{1.41}$$

where we introduced the space-time chirality operator $\Gamma^5 \equiv \sigma_3^{(0)} \otimes \sigma_3^{(1)}$. In a similar way, we introduce the chirality matrices in the spinor representation of the gauge groups $SO(14)$ and $SO(4)$, they are $\Gamma_{SO(14)} = \bigotimes_{l=1}^{7} \sigma_3^{(\bar{l})}$ and $\Gamma_{SO(4)} = \bigotimes_{l=15}^{16} \sigma_3^{(\bar{l})}$.

For any choice of the k_{ij}, the first equation tells us that the internal "spin" in space (5) is completely determined by the spacetime chirality. Both chiralities are possible, as they should be for a massive fermion. From the second equation we learn that the "spin" in space (4) is completely determined by the $U(1)$ charge of the particle; and finally, $\sigma_3^{(7)}$ and $\sigma_3^{(6)}$ are specified by the $SO(14)$ and $SO(4)$ chirality, respectively. Thus, for any choice of $U(1)$ charge and $SO(14)$ and $SO(4)$ chiralities, there exist two spin 1/2 fermions labelled by the eigenvalues of $\sigma_3^{(9)}$, ± 1.

Analogously, one can check that the projection conditions for the supersymmetric scalar partners of these fermions, in the $\mathbf{W}_{03}$ sector, leave only two free indices, the family index in the ninth space and the index in the eighth space which labels the two supersymmetric scalar partners of each massive fermion.

We are now in a position to understand how spacetime supersymmetry manifests itself in the spectrum.

A necessary (but not sufficient) condition for a generic KLT model to be spacetime supersymmetric is that, among the vectors describing the possible boundary conditions, there exists one, $\mathbf{W}_{\mathrm{SUSY}}$, where all components corresponding to fermions carrying gauge charges are zero, and the first right-moving component is 1/2. It is not hard to see why: If the model is supersymmetric, then for any state in some given sector $\boldsymbol{\alpha}$ there must exist an associated sector $\tilde{\boldsymbol{\alpha}}$ containing the superpartner state. If the original sector is fermionic ($\alpha_1 = 0$) the associated one is bosonic ($\tilde{\alpha}_1 = 1/2$) and vice versa; furthermore, if the two states are to have the same charges, it is necessary for the gauge charges of one sector to run over the same set of values as those of the other sector, that is, for all world-sheet fermions carrying gauge charges to have the same boundary conditions in the two sectors. Thus, $\mathbf{W}_{\mathrm{SUSY}} = \boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}$.

In our toy model $\mathbf{W}_{\mathrm{SUSY}}$ must have the form

$$\big((0)^{22}|(\tfrac{1}{2})(*\,*\,*)(*\,*\,*)(*\,*\,*)\big)\ . \tag{1.42}$$

There is only one such vector, namely

$$\mathbf{W}_{\mathrm{SUSY}} = \mathbf{W}_0 + \mathbf{W}_1 \ = \ \big((0)^{22}|(\tfrac{1}{2})(\tfrac{1}{2}00)^3\big)\ . \tag{1.43}$$

In conclusion, our toy model is spacetime supersymmetric if and only if equation (1.37) holds. And given a state in the supersymmetric model in the sector $m\mathbf{W}$, the superpartner resides in the sector $\mathbf{W}_0 + \mathbf{W}_1 + m\mathbf{W}$.

Notice that $\mathbf{W}_{\mathrm{SUSY}}$ also exchanges the boundary conditions of the internal world-sheet fermions $\psi_{(2)}$, $\psi_{(5)}$ and $\psi_{(8)}$. The associated degrees of freedom are not family indices for the states and should be considered instead as enumerative indices for the elements of the spacetime supermultiplets.

2. Amplitudes, Vertex Operators and Cocycles

In this section we introduce the tools necessary for the computation of arbitrary

amplitudes in a KLT string model. We will restrict ourselves to one-loop amplitudes but the generalization to higher loops is straightforward. For convenience we will also adopt the operator formalism, even if it is quite simple to express the following formulæ in terms of Polyakov path integrals.

We define the T-matrix element as the connected S-matrix element with certain normalization factors removed

$$\frac{\langle \lambda_1, \ldots, \lambda_{N_{\rm out}} | S_c | \lambda_{N_{\rm out}+1}, \ldots, \lambda_{N_{\rm out}+N_{\rm in}} \rangle}{\prod_{i=1}^{N_{\rm tot}} \left(\langle \lambda_i | \lambda_i \rangle \right)^{1/2}} = \tag{2.1}$$
$$i(2\pi)^4 \delta^4(p_1 + \ldots p_{N_{\rm tot}}) \prod_{i=1}^{N_{\rm tot}} (2E_i V)^{-1/2}\ T(\lambda_1, \ldots, \lambda_{N_{\rm out}} | \lambda_{N_{\rm out}+1}, \ldots, \lambda_{N_{\rm out}+N_{\rm in}})\ ,$$

where $N_{\rm tot} = N_{\rm in} + N_{\rm out}$ is the total number of external states. All momenta are oriented inwards so that a string state is considered ingoing (outgoing) if $p_i^0 > 0$ ($p_i^0 < 0$). $E_i = |p_i^0|$ is the energy of the i'th string state and V is the usual volume-of-the-world factor.

Corresponding to each state $|\lambda\rangle$ we have a vertex operator $\mathcal{V}_{|\lambda\rangle}(z, \bar{z})$ and the 1-loop contribution to the T-matrix element, $T^{1-{\rm loop}}$, is given by the operator formula

$$T^{1-{\rm loop}}(\lambda_1, \ldots, \lambda_{N_{\rm out}} | \lambda_{N_{\rm out}+1}, \ldots, \lambda_{N_{\rm out}+N_{\rm in}}) \ = \ C_{g=1} \int \prod_{I=1}^{N_{\rm tot}} {\rm d}^2 m^I \ \times \tag{2.2}$$
$$\sum_{m_i, n_j} C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \langle\langle \left| \prod_{I=1}^{N_{\rm tot}} (\eta_I | b) \prod_{i=1}^{N_{\rm tot}} c(z_i) \right|^2 \prod_{A=1}^{N_B + N_{FP}} \Pi(w_A)\ \mathcal{V}_{\langle \lambda_1 |}(z_1, \bar{z}_1) \ldots \mathcal{V}_{|\lambda_{N_{\rm tot}}\rangle}(z_{N_{\rm tot}}, \bar{z}_{N_{\rm tot}}) \rangle\rangle\ .$$

Here the constant $C_{g=1}$ gives the correct normalization of the vacuum amplitude. In $D = 4$ it is given by [17]

$$C_{g=1} = \left(\frac{1}{2\pi} \right)^2 (\alpha')^{-2}\ . \tag{2.3}$$

m^I is a modular parameter, η_I is the corresponding Beltrami differential [10], and the overlap $(\eta_I | b)$ with the antighost field b is given explicitly in ref. [23]. The integral is over one fundamental domain of N-punctured genus-one moduli space. By definition the correlator $\langle\langle \ldots \rangle\rangle$ includes the partition function (our conventions and normalizations for the partition function can be found in appendices A and B).

In an amplitude involving N_B space-time bosons and $2N_{FP}$ space-time fermions we have the insertion of $N_B + N_{FP}$ Picture Changing Operators (PCOs) $\Pi(w_A)$, given by eq. (2.13) below, at arbitrary points w_A on the Riemann surface. In practical

calculations we will always choose to insert one PCO at each of the vertex operators describing the space-time bosons. This leaves N_{FP} PCOs at arbitrary points.

In order to introduce explicitly the vertex operators it is convenient to bosonize all complex fermions according to

$$\begin{aligned} \psi_{(l)}(z) &= e^{\phi_{(l)}(z)} c_{(l)} & \psi^*_{(l)}(z) &= e^{-\phi_{(l)}(z)} c^*_{(l)} \\ \bar{\psi}_{(\bar{l})}(\bar{z}) &= e^{\bar{\phi}_{(\bar{l})}(\bar{z})} c_{(\bar{l})} & \bar{\psi}^*_{(\bar{l})}(\bar{z}) &= e^{-\bar{\phi}_{(\bar{l})}(\bar{z})} c^*_{(\bar{l})} \ , \end{aligned} \tag{2.4}$$

where the scalar field $\phi_{(l)}$ has operator product expansion (OPE)

$$\phi_{(l)}(z)\phi_{(k)}(w) = +\,\delta_{l,k} \log(z-w) + \dots \ . \tag{2.5}$$

The cocycle factors $c_{(l)}$ guarantee the correct anti-commutation relations between different fermions. We will return to them in the next subsection.

The ground state in the sector specified by α_l is created from the conformal vacuum by the spin field operator

$$S^{(l)}_{a_l}(z) = e^{a_l \phi_{(l)}(z)} (c_{(l)})^{a_l} \ , \tag{2.6}$$

with $a_l \in [-\frac{1}{2}; \frac{1}{2}]$ given by $\frac{1}{2} - \alpha_l$ mod 1. Notice that the R case ($\alpha_l = 0$) is unique in having two vacua, corresponding to $a_l = \pm 1/2$. We will sometimes use the abbreviation $S_\pm \equiv S_{\pm 1/2}$. An expression similar to (2.6) holds for the left-movers.

The scalar field is related to the fermion number current by

$$\partial \phi_{(l)} = -\psi^*_{(l)} \psi_{(l)} = -i \psi^1_{(l)} \psi^2_{(l)} \ , \tag{2.7}$$

and the corresponding number operators

$$J^{(l)}_0 = \oint_0 \frac{dz}{2\pi i} \partial \phi_{(l)}(z) \tag{2.8}$$

satisfy

$$[J^{(l)}_0, \phi_{(k)}] = \delta^l_k \tag{2.9}$$

and differ from the fermion number operators (1.11) only by a constant term:

$$J^{(l)}_0 = N^{(l)}_{[\![\alpha_l]\!]} + [\![1-\alpha_l]\!] - \tfrac{1}{2} \ . \tag{2.10}$$

We also "bosonize" the superghosts in the standard way

$$\begin{aligned} \beta &= \partial \xi \, e^{-\phi} (c_{(11)})^{-1} \\ \gamma &= e^{\phi} c_{(11)} \, \eta \ , \end{aligned} \tag{2.11}$$

where the scalar field ϕ has the "wrong" metric

$$\phi(z)\phi(w) = -\log(z-w) + \dots \ , \tag{2.12}$$

and $c_{(11)}$ is another cocycle factor. The PCO now assumes the form

$$\Pi = 2c\partial\xi + 2e^{\phi}c_{(11)}T_F^{[X,\psi]} - \tfrac{1}{2}\partial(e^{2\phi}(c_{(11)})^2\eta b) - \tfrac{1}{2}e^{2\phi}(c_{(11)})^2(\partial\eta)b \ . \tag{2.13}$$

If we define $\phi_{(11)} \equiv \phi$ the superghost part of any physical state vertex operator is given by eq. (2.6) for $l = 11$, with

$$a_{11} = -\tfrac{1}{2} - [\![\alpha_1]\!] = \begin{cases} -1 & \text{in bosonic sector} \\ -1/2 & \text{in fermionic sector} \end{cases} \ . \tag{2.14}$$

In any given sector $\boldsymbol{\alpha}$ the vertex operator describing the ground state of momentum p now assumes the form

$$\mathcal{V} = \mathcal{N} \cdot \prod_{\bar{l}=\bar{1}}^{\overline{22}} \bar{S}_{\bar{a}_l}^{(\bar{l})} \prod_{l=0}^{11} S_{a_l}^{(l)} \cdot e^{ik\cdot X} \equiv \mathcal{N} \cdot S_{\mathbb{A}} \cdot e^{ik\cdot X} \ , \tag{2.15}$$

where $\mathbb{A} \equiv (A; a_{11}) \equiv (\bar{a}_1, \dots, \bar{a}_{22}; a_0, a_1, \dots; a_{11})$ and we introduced the dimensionless momentum $k_\mu \equiv \sqrt{\frac{\alpha'}{2}}\, p_\mu$. The normalization constant $\mathcal{N}$ may be found using the method of ref. [17] (see also Appendix D).

Vertex operators describing excited states are constructed using the standard connection between mode operators and field operators. Physical external states are described by vertex operators $\mathcal{V}$ such that $\bar{c}c\mathcal{V}$ is BRST invariant. The BRST currents are given by

$$\begin{aligned} j_{\text{BRST}} &= cT_B^{[X,\psi,\beta,\gamma]} - cb\partial c - T_F^{[X,\psi]}e^{\phi}c_{(11)}\,\eta - \frac{1}{4}e^{2\phi}(c_{(11)})^2\,\eta(\partial\eta)b \\ \bar{\jmath}_{\text{BRST}} &= \bar{c}\bar{T}_B^{[\bar{X},\bar{\psi}]} - \bar{c}\bar{b}\bar{\partial}\bar{c} \ , \end{aligned} \tag{2.16}$$

where T_B and $\bar{T}_B$ are the energy-momentum tensors. The first-order pole in the OPE of $\bar{\jmath}_{\text{BRST}}$ with $\bar{c}c\mathcal{V}$, as well as the first order pole in the OPE of the first two terms of j_{BRST} with $\bar{c}c\mathcal{V}$, vanish merely by imposing that the vertex operator $\mathcal{V}$ is a primary conformal field of dimension one. In particular this implies that the string states satisfy the mass-shell condition $\bar{L}_0 = L_0 = 0$.

The last term in j_{BRST} has a non-singular OPE with $\bar{c}c\mathcal{V}$ for any operator $\mathcal{V}$ whose superghost part is given by $e^{-\phi}$ or $e^{-\phi/2}$. Therefore the BRST-invariance is reduced to the requirement that the first-order pole in the OPE $e^{\phi(w)}c_{(11)}T_F^{[X,\psi]}(w)\ c\bar{c}\mathcal{V}(z,\bar{z})$ should vanish. For a gauge boson, this equation becomes the transversality condition

$$\epsilon \cdot k \ = \ 0 \ , \tag{2.17}$$

whereas for a spacetime fermion it becomes the "Dirac equation", as it will be discussed in subsection §2.3 .

2.1 Choosing cocycles

In this subsection we consider in detail how to define the cocycle operators introduced by the bosonization (2.4).

The simplest example is the case of just two complex fermions, where we would define

$$c_{(1)} = \mathbf{1} \qquad \text{and} \qquad c_{(2)} = e^{\pm i\pi J_0^{(1)}} \ . \tag{2.18}$$

Clearly $\psi_{(1)}(z_1)\psi_{(2)}(z_2) = -\psi_{(2)}(z_2)\psi_{(1)}(z_1)$ regardless of which sign is chosen in the definition of $c_{(2)}$. But in the presence of spin fields the two choices of sign are no longer equivalent. Moving, say, $\psi_{(1)}$ through the spin field operator $S^{(2)}_{a_2}$ we pick up a phase $e^{\mp i\pi a_2}$. In general, when more than two fermions are involved, the cocycles involve a choice of many signs. But the various signs are not all independent: They have to be chosen in such a way that the left- and right-moving BRST currents have well-defined statistics with respect to all vertex operators, i.e. they are only allowed to pick up a possible *overall* phase when moved through a vertex operator. The relative signs between different terms should not change. Otherwise, a product of BRST invariant vertex operators would not necessarily be BRST invariant. Likewise, we must require that all Kač-Moody currents satisfy Bose statistics with respect to all vertex operators; otherwise, a product of vertex operators $\mathcal{V}_i$ transforming in various representations D_i of the gauge group would not necessarily transform in the tensor representation $\otimes_i D_i$. We also require that the PCO (2.13) should obey Bose statistics with respect to all vertex operators. In the present subsection we discuss how to make a consistent choice of cocycles, and we present an explicit solution in the case of our toy model. The discussion generalizes that of ref. [15].

We write the cocycle operators as follows

$$\begin{aligned} c_{(\bar l)} &= c_{\rm gh}^{(\bar l)} \cdot \exp\left\{ i\pi \sum_{j=1}^{l-1} Y_{\bar l \bar \jmath} \bar J_0^{(\bar \jmath)} \right\} \qquad \text{for} \qquad \bar l = \bar 1, \ldots, \overline{22} \\ c_{(l)} &= c_{\rm gh}^{(l)} \cdot \exp\left\{ i\pi \left(\sum_{j=1}^{22} Y_{l\bar \jmath} \bar J_0^{(\bar \jmath)} + \sum_{j=0}^{l-1} Y_{lj} J_0^{(j)} \right) \right\} \qquad \text{for} \qquad l = 0, 1, \ldots, 10, 11 \ , \end{aligned} \tag{2.19}$$

with

$$\begin{aligned} c_{\rm gh}^{(l)} &\equiv \exp\{-i\pi\varepsilon^{(l)} N_{(\eta,\xi)}\} \exp\{i\pi\varepsilon^{(l)}(N_{(b,c)} - N_{(\bar b,\bar c)})\} \\ c_{\rm gh}^{(\bar l)} &\equiv \exp\{-i\pi\varepsilon^{(\bar l)} N_{(\eta,\xi)}\} \exp\{i\pi\varepsilon^{(\bar l)}(N_{(b,c)} - N_{(\bar b,\bar c)})\} \ . \end{aligned} \tag{2.20}$$

Here all the parameters Y, as well as the ε, take values either $+1$ or -1, and $N_{(b,c)}$, $N_{(\bar{b},\bar{c})}$ and $N_{(\eta,\xi)}$ are the number operators of the (b,c), $(\bar{b},\bar{c})$ and (η,ξ) systems respectively. The form chosen for $c_{\rm gh}^{(l)}$ and $c_{\rm gh}^{(\bar{l})}$ is one of convenience: It ensures that the first term, $2c\partial\xi$, in the PCO, as well as operators like $\bar{b}b$ and $\bar{c}c$, commute with any spin field operator (2.6).

It is convenient to introduce a more compact notation: Let capital indices K and L run over the set of values $\{\bar{1},\ldots,\overline{22};0,1,\ldots,10,11\}$. Define

$$\Phi_{(L)} = \begin{cases} \bar{\phi}_{(\bar{l})} & \text{for} \qquad L=\bar{l}=\bar{1},\ldots,\overline{22} \\ \phi_{(l)} & \text{for} \quad L=l=0,1,\ldots,10,11 \end{cases} . \tag{2.21}$$

We may then recast the definitions (2.19) on the form

$$C_{(L)} = C_{\rm gh}^{(L)} \cdot e^{i\pi e_{(L)}\cdot Y\cdot J_0} , \tag{2.22}$$

where the $(\overline{22}|12)\times(\overline{22}|12)$ matrix

$$Y_{LL'} = \begin{bmatrix} Y_{\bar{l}\bar{l}'} & 0 \\ Y_{l\bar{l}'} & Y_{ll'} \end{bmatrix} \tag{2.23}$$

is lower triangular, J_0 is the $(\overline{22}|12)$ vector of number operators (2.8) and $e_{(L)}$ is the unit vector with components $(e_{(L)})_K = \delta_{L,K}$.

From the definitions (2.4), (2.6) and (2.19) one finds for $K\neq L$

$$\Psi_{(L)}(z,\bar{z})S_{\mathbb{A}_K}^{(K)}(w,\bar{w}) = S_{\mathbb{A}_K}^{(K)}(w,\bar{w})\Psi_{(L)}(z,\bar{z})e^{-i\pi Y_{KL}\mathbb{A}_L + i\pi Y_{LK}\mathbb{A}_K} , \tag{2.24}$$

where we introduced the rather obvious notation

$$\Psi_{(L)} = e^{\Phi_{(L)}}C_{(L)} \qquad \text{and} \qquad S_{\mathbb{A}_L}^{(L)} = e^{\mathbb{A}_L\Phi_{(L)}}(C_{(L)})^{\mathbb{A}_L} . \tag{2.25}$$

In order to generalize eq. (2.24) to the case $L=K$ we study the branch cut behaviour present in the OPE:

$$\begin{aligned} \psi_{(l)}(z)S_{a_l}^{(l)}(w) &= (z-w)^{a_l}e^{(1+a_l)\phi^{(l)}(w)}+\ldots \qquad \text{for} \qquad l=0,1,\ldots,10 \\ \bar{\psi}_{(\bar{l})}(\bar{z})\bar{S}_{\bar{a}_l}^{(\bar{l})}(\bar{w}) &= (\bar{z}-\bar{w})^{\bar{a}_l}e^{(1+\bar{a}_l)\bar{\phi}^{(\bar{l})}(\bar{w})}+\ldots \qquad \text{for} \qquad \bar{l}=\bar{1},\ldots,\overline{22} \\ e^{\phi(z)}e^{a_{11}\phi(w)} &= (z-w)^{-a_{11}}e^{(1+a_{11})\phi(w)}+\ldots . \end{aligned} \tag{2.26}$$

If we make the phase choice

$$\left(\frac{z-w}{w-z}\right) = e^{\varepsilon i\pi} \qquad , \qquad \varepsilon=\pm 1 , \tag{2.27}$$

eqs. (2.24) and (2.26) may be summarized in a single equation

$$\Psi_{(L)}(z,\bar{z})S^{(K)}_{\mathbb{A}_K}(w,\bar{w}) = S^{(K)}_{\mathbb{A}_K}(w,\bar{w})\Psi_{(L)}(z,\bar{z})e^{i\pi\tilde{Y}_{LK}\mathbb{A}_K} \ , \tag{2.28}$$

where we introduced the matrix $\tilde{Y}$ obtained by anti-symmetrizing the lower-triangular matrix, Y, and adding the diagonal elements

$$\tilde{Y}_{LL} = \begin{cases} -\varepsilon & \text{for} \quad L = \bar{l} = \bar{1}, \ldots, \overline{22} \\ \varepsilon & \text{for} \quad L = l = 0, 1, \ldots, 10 \\ -\varepsilon & \text{for} \quad L = 11 \end{cases} . \tag{2.29}$$

In order to ensure that all spin field operators commute with β and γ we take

$$\varepsilon^{(L)} = \tilde{Y}_{11,L} \ . \tag{2.30}$$

For a generic product of spin fields, as defined in eq. (2.15) we find by repeated use of (2.28)

$$\Psi_{(L)}(z,\bar{z})S_{\mathbb{A}}(w,\bar{w}) = S_{\mathbb{A}}(w,\bar{w})\Psi_{(L)}(z,\bar{z})e^{i\pi\varphi_L[\mathbb{A}]} \ , \tag{2.31}$$

with

$$\varphi_L[\mathbb{A}] \equiv \sum_K \tilde{Y}_{LK}\mathbb{A}_K \ \text{mod} \ 2 \ , \tag{2.32}$$

while $\Psi^*_{(L)}(z,\bar{z})$ picks up the complex conjugate phase.

We are now in a position to investigate the constraints on the matrix $\tilde{Y}$ imposed by the requirement that the BRST current should have well-defined statistics with respect to all vertex operators. Since all raising (and lowering) operators do have well-defined statistics it is sufficient to consider just the ground state vertex operators. These are given by eq. (2.15) with

$$\mathbb{A}_L = \tfrac{1}{2} - \sum_i m_i(\mathbf{V}_i)_{(L)} + \mathbb{N}_L \ , \tag{2.33}$$

where $\mathbf{V}_i$ is the $(\overline{22}|12)$ vector obtained from $\mathbf{W}_i$ by adding the components $(\mathbf{V}_i)_{(0)}$ and $(\mathbf{V}_i)_{(11)}$ given in the obvious way by

$$(\mathbf{V}_i)_{(0)} = (\mathbf{V}_i)_{(11)} = (\mathbf{V}_i)_{(1)} = (\mathbf{W}_i)_{(1)} = s_i \ , \tag{2.34}$$

and $\mathbb{N}_L$ is a set of appropriate integers. Now we see that whereas the left-moving BRST current has well-defined statistics with respect to $S_{\mathbb{A}}$,

$$\bar{\jmath}_{BRST}(\bar{z})S_{\mathbb{A}}(w,\bar{w}) = S_{\mathbb{A}}(w,\bar{w})\bar{\jmath}_{BRST}(\bar{z})e^{i\pi\varphi_{11}[\mathbb{A}]} \ , \tag{2.35}$$

the different terms in the right-moving BRST current will in general not pick up the same phase. Particularly non-trivial is the requirement that all terms in the supercurrent (1.3) pick up the same phase.

First of all, in order for the real fermions $\psi^m_{(l)}, l = 0, 1, \ldots, 10$ to have well-defined statistics we need to have

$$\varphi_l[\mathbb{A}] = \text{integer} \qquad \text{for} \quad l = 0, 1, \ldots, 10 \ . \tag{2.36}$$

Since by the constraint (1.6) the number of right-moving fermions having R boundary conditions (and hence half-integer a_l) is even, eq. (2.36) is actually equivalent to

$$\sum_{k=1}^{22} \tilde{Y}_{l\bar{k}} \bar{a}_k = \text{integer} \qquad \text{for} \quad l = 0, 1, \ldots, 10 \ . \tag{2.37}$$

If we further require

$$\varphi_0[\mathbb{A}] \overset{\text{MOD } 2}{=} \varphi_1[\mathbb{A}] \overset{\text{MOD } 2}{=} \sum_{k=2,3,4} \varphi_k[\mathbb{A}] \overset{\text{MOD } 2}{=} \sum_{k=5,6,7} \varphi_k[\mathbb{A}] \overset{\text{MOD } 2}{=} \sum_{k=8,9,10} \varphi_k[\mathbb{A}] \ , \tag{2.38}$$

we find

$$T_F^{[X,\psi]}(z) S_{\mathbb{A}}(w, \bar{w}) = S_{\mathbb{A}}(w, \bar{w}) T_F^{[X,\psi]}(z) e^{i\pi\varphi_1[\mathbb{A}]} \ . \tag{2.39}$$

If we also require

$$\varphi_{11}[\mathbb{A}] \overset{\text{MOD } 2}{=} \varphi_1[\mathbb{A}] \ , \tag{2.40}$$

the entire BRST current j_{BRST} will satisfy

$$j_{BRST}(z) S_{\mathbb{A}}(w, \bar{w}) = S_{\mathbb{A}}(w, \bar{w}) j_{BRST}(z) e^{i\pi\varphi_1[\mathbb{A}]} = \pm S_{\mathbb{A}}(w, \bar{w}) j_{BRST}(z) \ , \tag{2.41}$$

and the PCO (2.13) will pick up no phase at all

$$\Pi(z) S_{\mathbb{A}}(w, \bar{w}) = S_{\mathbb{A}}(w, \bar{w}) \Pi(z) \ . \tag{2.42}$$

The constraints (2.36), (2.38) and (2.40) should be satisfied for all sectors. If we insert the value (2.33) and define the $(\overline{22}|12)$ vectors

$$(\tilde{\mathbf{V}}_i)_{(L)} \equiv \tfrac{1}{2} \sum_K \tilde{Y}_{LK} (\mathbf{V}_i)_{(K)} \ , \tag{2.43}$$

the constraints (2.36), (2.38) and (2.40) are seen to be equivalent to

$$\begin{aligned}
&(\tilde{\mathbf{V}}_i)_{(0)} \stackrel{\text{MOD }1}{=} (\tilde{\mathbf{V}}_i)_{(1)} \stackrel{\text{MOD }1}{=} \\
&\sum_{k=2,3,4} (\tilde{\mathbf{V}}_i)_{(k)} \stackrel{\text{MOD }1}{=} \sum_{k=5,6,7} (\tilde{\mathbf{V}}_i)_{(k)} \stackrel{\text{MOD }1}{=} \sum_{k=8,9,10} (\tilde{\mathbf{V}}_i)_{(k)} \stackrel{\text{MOD }1}{=} (\tilde{\mathbf{V}}_i)_{(11)}
\end{aligned} \tag{2.44}$$

and

$$2(\tilde{\mathbf{V}}_i)_{(1)} \stackrel{\text{MOD }1}{=} 0 \;, \tag{2.45}$$

regardless of the values of the integers $\mathbb{N}_L$. That is, the right-moving components of the vectors $\tilde{\mathbf{V}}_i$ should satisfy exactly the same properties as the right-moving components of the vectors $\mathbf{V}_i$.

The requirement that all Kač-Moody currents should have Bose statistics with respect to $S_{\mathbb{A}}$ places further constraints on the matrix $\tilde{Y}$; these constraints will also involve the left-moving components of $\tilde{\mathbf{V}}_i$. To be more explicit we consider our toy model.

2.2 Choosing cocycles in the toy model

In the toy model described in subsection 1.2 the Kač-Moody currents corresponding to the gauge group (1.29) are given by $\bar{\psi}^m_{(\bar{l})}\bar{\psi}^n_{(\bar{k})}(\bar{z})$, where $m,n=1,2$ and $\bar{l}$ and $\bar{k}$ *both* belong to *one* of the five subsets $\{\bar{1},\ldots,\bar{7}\}$, $\{\bar{8},\ldots,\overline{14}\}$, $\{\overline{15},\overline{16}\}$, $\{\overline{17}\}$, $\{\overline{18},\ldots,\overline{22}\}$.

In order for these currents to have Bose statistics with respect to any operator $S_{\mathbb{A}}$ we need the phases $\varphi_{\bar{l}}[\mathbb{A}]$ to be integer, and to assume always the same value (mod 2) for any value of $\bar{l}$ inside one of the above subsets. This translates into the requirement that the left-moving components of the vectors $\tilde{\mathbf{V}}_i$ should be either integer or half-integer and satisfy

$$\begin{aligned}
&(\tilde{\mathbf{V}}_i)_{(\bar{1})} \stackrel{\text{MOD }1}{=} (\tilde{\mathbf{V}}_i)_{(\bar{2})} \stackrel{\text{MOD }1}{=} \ldots \stackrel{\text{MOD }1}{=} (\tilde{\mathbf{V}}_i)_{(\bar{7})} \\
&(\tilde{\mathbf{V}}_i)_{(\bar{8})} \stackrel{\text{MOD }1}{=} \ldots \stackrel{\text{MOD }1}{=} (\tilde{\mathbf{V}}_i)_{(\overline{14})} \\
&(\tilde{\mathbf{V}}_i)_{(\overline{15})} \stackrel{\text{MOD }1}{=} (\tilde{\mathbf{V}}_i)_{(\overline{16})} \\
&(\tilde{\mathbf{V}}_i)_{(\overline{18})} \stackrel{\text{MOD }1}{=} \ldots \stackrel{\text{MOD }1}{=} (\tilde{\mathbf{V}}_i)_{(\overline{22})} \;.
\end{aligned} \tag{2.46}$$

In any model based on vectors $\mathbf{W}_i$ which have only 0 and 1/2 entries (so that all $M_i=2$) the phases $\varphi_K[\mathbb{A}]$ are guaranteed to be integer, since all $\mathbb{A}_K$ are either integer or half-integer and the number of $\mathbb{A}_K$ that are half-integer is always even. The latter statement

follows from the conditions (1.14) which imply that $\mathbf{W}_i \cdot \mathbf{W}_i = 2k_{ii}$ is an integer, so that the vector $\mathbf{W}_i$ contains an even number of components that are 1/2 (and hence also an even number that are 0). This property is inherited by any vector $\sum_i m_i \mathbf{W}_i$.

The constraints (2.44) and (2.46) allow many choices for the cocycle matrix $Y_{LL'}$ since the number of free variables far exceeds the number of constraints. A convenient choice is to take [1]

$$
Y = \begin{bmatrix}
Y_{7,7} & & & & & & & & & \\
\mathbf{1}_{7,7} & Y_{7,7} & & & & & & & & \\
\mathbf{1}_{2,7} & \mathbf{1}_{2,7} & \mathbf{1}^{\mathrm{LT}}_{2,2} & & & & & & & \\
\mathbf{1}_{1,7} & \mathbf{1}_{1,7} & \mathbf{1}_{1,2} & 0 & & & & & & \\
\mathbf{1}_{5,7} & \mathbf{1}_{5,7} & \mathbf{1}_{5,2} & \mathbf{1}_{5,1} & Y_{5,5} & & & & & \\
\mathbf{1}_{2,7} & \mathbf{1}_{2,7} & \mathbf{1}_{2,2} & \mathbf{1}_{2,1} & \mathbf{1}_{2,5} & \mathbf{1}^{\mathrm{LT}}_{2,2} & & & & \\
\mathbf{1}_{3,7} & \mathbf{1}_{3,7} & \mathbf{1}_{3,2} & \mathbf{1}_{3,1} & \mathbf{1}_{3,5} & \mathbf{1}_{3,2} & Y_{3,3} & & & \\
\mathbf{1}_{3,7} & \mathbf{1}_{3,7} & Y_{3,2} & \mathbf{1}_{3,1} & \mathbf{1}_{3,5} & \mathbf{1}_{3,2} & \mathbf{1}_{3,3} & \mathbf{1}^{\mathrm{LT}}_{3,3} & & \\
\mathbf{1}_{3,7} & \mathbf{1}_{3,7} & \mathbf{1}_{3,2} & \mathbf{1}_{3,1} & \mathbf{1}_{3,5} & \mathbf{1}_{3,2} & \mathbf{1}_{3,3} & \mathbf{1}_{3,3} & Y_{3,3} & \\
-\mathbf{1}_{1,7} & -\mathbf{1}_{1,7} & -\mathbf{1}_{1,2} & -1 & -\mathbf{1}_{1,5} & 1\ -1 & -\mathbf{1}_{1,3} & -\mathbf{1}_{1,3} & -\mathbf{1}_{1,3} & 0
\end{bmatrix}, \quad (2.47)
$$

where $\mathbf{1}_{m,n}$ is the $m \times n$ matrix with all elements equal to 1; $\mathbf{1}^{\mathrm{LT}}_{m,m}$ is the $m \times m$ matrix which has all elements 1 in the lower triangle and the rest equal to zero; and

$$
Y_{7,7} = \begin{pmatrix}
0 & & & & & & \\
1 & 0 & & & & & \\
1 & 1 & 0 & & & & \\
1 & 1 & 1 & 0 & & & \\
1 & 1 & 1 & 1 & 0 & & \\
1 & 1 & 1 & 1 & 1 & 0 & \\
-1 & 1 & -1 & 1 & -1 & 1 & 0
\end{pmatrix} \quad (2.48)
$$

$$
Y_{5,5} = \begin{pmatrix}
0 & & & & \\
1 & 0 & & & \\
-1 & -1 & 0 & & \\
1 & 1 & -1 & 0 & \\
-1 & 1 & -1 & 1 & 0
\end{pmatrix} \quad (2.49)
$$

$$
Y_{3,3} = \begin{pmatrix}
0 & & \\
1 & 0 & \\
-1 & 1 & 0
\end{pmatrix} \quad (2.50)
$$

$$
Y_{3,2} = \begin{pmatrix}
1 & 1 \\
-1 & 1 \\
1 & 1
\end{pmatrix}. \quad (2.51)
$$

[1] For these matrices we adopt the convention that where there is a missing entry, one should put zero.

We define the set of 4-dimensional gamma matrices by means of the OPE between the real space-time fermions ψ^μ and the space-time spin field $S_\alpha \equiv S^{(0)}_{a_0} S^{(1)}_{a_1}$:

$$\psi^\mu(z) S_\alpha(w) \stackrel{\text{OPE}}{=} \frac{1}{\sqrt{2}} (\Gamma^\mu)_\alpha^{\ \beta} \frac{S_\beta(w)}{\sqrt{z-w}} \ . \tag{2.52}$$

Then, corresponding to each of the two possible choices for Y_{10} we have an explicit representation of the gamma matrices. The cocycle choice (2.47) (which has $Y_{10} = +1$) gives rise to

$$\begin{aligned} \Gamma^0 &= \sigma_1^{(0)} \otimes \sigma_0^{(1)} \\ \Gamma^1 &= -\sigma_2^{(0)} \otimes \sigma_0^{(1)} \\ \Gamma^2 &= \sigma_3^{(0)} \otimes \sigma_2^{(1)} \\ \Gamma^3 &= \sigma_3^{(0)} \otimes \sigma_1^{(1)} \ , \end{aligned} \tag{2.53}$$

where σ_0 denotes the 2×2 unit matrix. Choosing instead $Y_{10} = -1$ would change the sign of Γ^2 and Γ^3. Notice that this way of defining the gamma matrices makes no reference to the choice of model or the particular string state we happen to consider. However, it has its own drawbacks. Indeed, in computing amplitudes involving vertex operators like (2.15), the gamma matrices arise from an operator product expansion like $\psi^\mu(z) S_{\mathbb{A}}(w, \bar{w})$. Obviously, in moving $\psi^\mu(z)$ across all the left-moving spin fields, using eq. (2.28), one will acquire the phase factor

$$\exp\{i\pi \sum_{l=1}^{22} \tilde{Y}_{s\bar{l}} \bar{a}_l\} \ , \tag{2.54}$$

where $s = 0$ for $\mu = 0, 1$ and $s = 1$ for $\mu = 2, 3$. By eq. (2.37) this phase factor is just a sign, but it could still differ in the two cases $s = 0$ and $s = 1$. In this case, to have a Lorentz covariant formulation, one would need to redefine, say, the gamma matrices Γ^2 and Γ^3 by a sign as compared to (2.53). To avoid this, one can make a cocycles' choice such that

$$\sum_{l=1}^{22} (Y_{0\bar{l}} - Y_{1\bar{l}}) \bar{a}_l \stackrel{\text{MOD } 2}{=} 0 \ . \tag{2.55}$$

This condition is trivially satisfied by our cocycles' choice eq. (2.47).

2.3 Vertex operators in the toy model

In this subsection we will introduce the vertex operators necessary for the computation of the amplitude we have chosen to consider: The one-loop three-point amplitude of a

"photon" (that is, the $U(1)$ gauge boson) and two massive charged fermions. We choose to consider the $\alpha' M^2 = 1$ spacetime fermions that form the ground states in the $\mathbf{W}_{13}$ sector. They have nonzero $U(1)$ charge and belong to a $(\frac{1}{2}, 0)$ multiplet when the model is spacetime supersymmetric. We call them "electrons" ("positrons") depending on whether the $U(1)$ charge is negative (positive). Obviously these names should not be taken too literally.

The vertex operator for the photon is given by [14,17]:

$$\mathcal{V}^{(-1)}_{\text{photon}}(z, \bar{z}; k; \epsilon) = \frac{\kappa}{\pi} \bar{\psi}_{(\overline{17})} \bar{\psi}^*_{(\overline{17})}(\bar{z})\ \epsilon \cdot \psi(z)\, e^{-\phi(z)}\, (c_{(11)})^{-1} e^{ik\cdot X(z,\bar{z})}\ , \tag{2.56}$$

where $\epsilon \cdot \epsilon = 1$ and the gravitational coupling κ is related to Newton's constant by $\kappa^2 = 8\pi G_N$. Here the label (-1) specifies the superghost charge of the vertex (i.e. the "picture"). For future convenience we also give the once picture-changed version of this vertex

$$\begin{aligned}\mathcal{V}^{(0)}_{\text{photon}}(z, \bar{z}; k; \epsilon) &= \lim_{w\to z} \Pi(w)\, \mathcal{V}^{(-1)}_{\text{photon}}(z, \bar{z}; k; \epsilon) = \\ &- i\frac{\kappa}{\pi} \bar{\psi}_{(\overline{17})} \bar{\psi}^*_{(\overline{17})}(\bar{z})\ \left[\epsilon \cdot \partial_z X(z) - ik \cdot \psi(z) \epsilon \cdot \psi(z)\right]\, e^{ik\cdot X(z,\bar{z})}\ ,\end{aligned} \tag{2.57}$$

where $k^2 = \epsilon \cdot k = 0$.

The vertex operator for the electron/positron is given by

$$\mathcal{V}^{(-1/2)}(z, \bar{z}; k; \mathbb{V}) \ = \ N_f\ \overline{\mathbf{V}}^{\bar{a}} \bar{S}_{\bar{a}}(\bar{z}) \times \mathbf{V}^a\ S_a(z)\ e^{-\frac{1}{2}\phi(z)} (c_{(11)})^{-1/2}\ e^{ik\cdot X(z,\bar{z})}\ , \tag{2.58}$$

where

$$\bar{S}_{\bar{a}}(\bar{z}) \equiv \prod_{l=1}^{7} \prod_{l=15}^{17} \bar{S}^{(\bar{l})}_{\bar{a}_l}(\bar{z}) \qquad \text{and} \qquad S_a(z) \equiv \prod_{l=0,1,4,5,6,7,9} S^{(l)}_{a_l}(z)\ . \tag{2.59}$$

The normalization constant N_f is computed in appendix D . The left-moving spinor indices $\bar{a} = \{\bar{a}_1, \ldots, \bar{a}_7; \bar{a}_{15}, \bar{a}_{16}; \bar{a}_{17}\}$ all take values $\pm 1/2$ and indicate that the fermion transforms in the spinor representation of the first $SO(14)$ and of the $SO(4)$, and $\bar{a}_{17} = \pm\frac{1}{2}$ is the $U(1)$ charge. The right-moving spinor index $a = (\alpha; a_{int}) = (a_0, a_1; a_4, a_5, a_6, a_7, a_9)$ also takes values $\pm 1/2$ and consists of the 4-dimensional space-time spinor index α as well as family and enumerative indices.

The spinor decomposes accordingly

$$\mathbb{V}^A = \overline{\mathbf{V}}^{\bar{a}}\ \mathbf{V}^a\ , \tag{2.60}$$

where

$$\overline{\mathbf{V}}^{\bar{a}} = \bar{V}^{\bar{a}_1,\ldots,\bar{a}_7}_{SO(14)}\ \bar{V}^{\bar{a}_{15},\bar{a}_{16}}_{SO(4)}\ \bar{V}^{\bar{a}_{17}}_{U(1)}\ , \tag{2.61}$$

and

$$\mathbf{V}^a = V^\alpha v^{a_{int}} = V^\alpha \prod_{l=4,5,6,7,9} v^{a_l}_{(l)} \tag{2.62}$$

is the product of the space-time spinor V^α and the two-dimensional "internal" spinors $v^{a_l}_{(l)}$. The right-moving spinor $\mathbf{V}$ satisfies a "Dirac equation", which (as explained at the beginning of section §2) is obtained from the requirement that the single pole in the OPE of $e^\phi T_F^{[X,\psi]}$ with the vertex operator (2.58) should vanish. One finds

$$\begin{aligned}
&\mathbf{V}^T(k)(\not{k} + \mathbf{M}) \;=\; 0 \qquad\qquad (2.63)\\
&\mathbf{M} \equiv -\frac{1}{2}\Gamma^5 \otimes \sigma_3^{(4)} \otimes \left(\sigma_1^{(5)} \otimes \sigma_1^{(6)} \otimes \sigma_1^{(7)} + \sigma_2^{(5)} \otimes \sigma_2^{(6)} \otimes \sigma_2^{(7)}\right) \otimes \sigma_0^{(9)} \; .
\end{aligned}$$

In this formula $\sigma_m^{(l)}$ for $m = 1, 2, 3$ are the Pauli matrices acting in the (l) space whereas $\sigma_0^{(l)}$ is the two dimensional identity matrix acting on the (l) space. The overall sign of the "mass operator" $\mathbf{M}$ depends on the cocycle choice. The sign quoted in (2.63) corresponds to the choice (2.47).

For a generic ground state (2.15) it is convenient to define a "generalized charge conjugation matrix" $\mathbb{C}$ by

$$S_{\mathbb{A}}(z, \bar{z}) S_{\mathbb{B}}(w, \bar{w}) \stackrel{\text{OPE}}{=} \mathbb{C}_{AB} \delta_{a_{11}, b_{11}} \; \frac{1}{(z-w)^p} \frac{1}{(\bar{z}-\bar{w})^{\bar{p}}} \; , \tag{2.64}$$

where $p = \sum_{l=0}^{11} (a_l)^2$ and $\bar{p} = \sum_{l=1}^{22} (\bar{a}_l)^2$. This matrix is related to the choice of cocycles by

$$\mathbb{C}_{AB} = e^{i\pi \mathbb{A}\cdot Y \cdot \mathbb{B}} \delta_{A+B} \; , \tag{2.65}$$

where it is understood that $a_{11} = b_{11}$ is given by (2.14). In the case of the electron/positron (2.58) the cocycle choice (2.47) leads to [2]

$$\mathbb{C}_{AB} = i e^{i\pi\varphi_c} \left(\mathrm{C}_{SO(14)} \otimes \mathrm{C}_{SO(4)} \otimes \sigma_1^{(\overline{17})}\right)_{\bar{a}\bar{b}} \; \mathbf{C}_{ab} \; , \tag{2.66}$$

where

$$\mathrm{C}_{SO(14)} = \sigma_2^{(\bar{1})} \otimes \sigma_1^{(\bar{2})} \otimes \sigma_2^{(\bar{3})} \otimes \sigma_1^{(\bar{4})} \otimes \sigma_2^{(\bar{5})} \otimes \sigma_1^{(\bar{6})} \otimes \sigma_2^{(\bar{7})} \tag{2.67}$$

and

$$\mathrm{C}_{SO(4)} = \sigma_2^{(\overline{15})} \otimes \sigma_1^{(\overline{16})} \tag{2.68}$$

---

[2] We hope that the reader will not be confused by C the conjugation matrices, $C_{g=1}$ the normalization of the amplitude as in eq. (2.3), and $C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}$ the phase coefficients of the sum over the spin structures given by eq. (1.17).

are standard spinor metrics; and

$$\mathbf{C}_{ab} = \mathrm{C} \otimes \sigma_1^{(4)} \otimes \sigma_1^{(5)} \otimes \sigma_2^{(6)} \otimes \sigma_1^{(7)} \otimes \sigma_1^{(9)} \ , \tag{2.69}$$

where

$$\mathrm{C} = \ -i\, \sigma_2^{(0)} \otimes \sigma_1^{(1)} \tag{2.70}$$

is the standard charge conjugation matrix which satisfies

$$\Gamma^\mu \mathrm{C} \ = \ -\mathrm{C}\, (\Gamma^\mu)^T \qquad\qquad C^T = C^{-1} = -C \ . \tag{2.71}$$

The phase appearing in (2.66) depends on whether $\bar{b}_1 + \ldots + \bar{b}_7 + \bar{b}_{15} + \bar{b}_{16} + \bar{b}_{17} + b_0 + b_1 + b_4 + \ldots + b_7 + b_9 - 1/2$ is an even or an odd integer. When $\mathbb{C}$ acts on the spinor $\mathbb{V}$ this is in turn determined by the GSO projections (1.41). One finds

$$\varphi_{\mathrm{c}} = +3/4 + [\![k_{00} + k_{01} + k_{03}]\!] \ . \tag{2.72}$$

Notice that the charge conjugation matrix satisfies

$$\mathbf{MC} \ = \ \mathbf{CM}^T \ . \tag{2.73}$$

For a generic choice of cocycles, the precise form of $\mathbb{C}$ can change, but one may verify that any choice of cocycles consistent with the constraints (2.38) leads to a matrix $\mathbf{C}$ such that

$$\Gamma^\mu \mathrm{C} \ = \ -\eta\, \mathrm{C}\, (\Gamma^\mu)^T \qquad\qquad \mathbf{MC} \ = \ \eta\, \mathbf{CM}^T \ , \tag{2.74}$$

with $\eta = \pm 1$, so that the Dirac equation (2.63) is equivalent to

$$(\not{k} - \mathbf{M})\mathbf{CV}(k) = 0 \ . \tag{2.75}$$

The choice $\eta = +1$ is preferable since only then is C the standard charge-conjugation matrix, but in what follows we will only need eq. (2.75).

3. A Sample Calculation: The Anomalous Magnetic Moment at 1 Loop

In this subsection we perform the explicit computation of a 1-loop amplitude with external space-time fermions. It is convenient to demonstrate how the machinery works in a simple example, the procedure for any other 1-loop amplitude being completely analogous.

For the reasons explained in the introduction, we have chosen to compute the three point amplitude of one photon (2.56) and two "electrons/positrons" (2.58), i.e. we consider, say, the process

$$e^{\pm} \;\rightarrow\; e^{\pm} \;+\; \gamma\ . \tag{3.1}$$

The 1-loop T-matrix element for this process is given by eq. (2.2):

$$\begin{aligned}
&T^{1-\text{loop}}(\ e^{\pm} \;\rightarrow\; e^{\pm} \;+\; \gamma) \;= \\
&C_{g=1}\int \mathrm{d}^2\tau \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 \sum_{m_i,n_j} C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\ \langle\langle\,|(\eta_\tau|b)(\eta_{z_1}|b)(\eta_{z_2}|b)\,c(z)c(z_1)c(z_2)|^2 \times \\
&\Pi(w_1)\ \Pi(w_2)\ \mathcal{V}^{(-1)}_{\text{photon}}(z,\bar z;k;\epsilon)\ \mathcal{V}^{(-1/2)}(z_1,\bar z_1;k_1;\mathbb{V}_1)\ \mathcal{V}^{(-1/2)}(z_2,\bar z_2;k_2;\mathbb{V}_2)\,\rangle\rangle\ ,
\end{aligned} \tag{3.2}$$

where we used translational invariance of the torus to fix the position z of the photon vertex operator at an arbitrary value.

We would like also to stress that the computation will be done without explicitly using the cocycles' choice eq. (2.47), we will only need eq. (2.75) which follows from the general properties of the cocycles as discussed in the previous section.

Before turning to the actual computation of the correlation functions appearing in (3.2) we would like to make a few observations.

3.1 Decomposition in Lorentz structures

As it is obvious from Lorentz covariance and from the spacetime structure of the expression (3.2) the amplitude must have an on-shell Lorentz decomposition which can be written as follows

$$\begin{aligned}
T^{1-\text{loop}}(\ e^{\pm} \;\rightarrow\; e^{\pm} \;+\; \gamma) \;=\; &\epsilon_\mu \mathbf{V}_1^T \mathbf{M}^2\,\Gamma^\mu \mathbf{C}\,\mathbf{V}_2\,T^{1-\text{loop}}_{\text{REN}} + \\
&\epsilon_\mu k_\nu \mathbf{V}_1^T \mathbf{M}\,\Gamma^{\mu\nu}\mathbf{C}\,\mathbf{V}_2\,T^{1-\text{loop}}_{\text{AMM}} + \epsilon_\mu k_\nu \mathbf{V}_1^T \mathbf{M}\,\Gamma^{\mu\nu}\Gamma^5 \mathbf{C}\,\mathbf{V}_2\,T^{1-\text{loop}}_{\text{PEDM}}\ .
\end{aligned} \tag{3.3}$$

Here the first term has the same structure as the tree-level amplitude and will be absorbed by a combination of vertex, wave-function and mass renormalization. The second and third terms contribute to the Anomalous Magnetic Moment and the Pseudo-Electric Dipole moment respectively. Our aim is to compute these two contributions, $T^{1-\text{loop}}_{\text{AMM}}$ and $T^{1-\text{loop}}_{\text{PEDM}}$, and discuss when they vanish.

To arrive at the decomposition (3.3) is actually quite non-trivial in our bosonized approach. As is clear from the OPE (2.52) each factor ψ^μ appearing in (3.2) should give

rise to a gamma matrix. However, the gamma matrices only appear after all the cocycle algebra has been performed. Furthermore, Lorentz covariance requires that the quantities $T_{\text{REN}}^{1-\text{loop}}$, $T_{\text{AMM}}^{1-\text{loop}}$ and $T_{\text{PEDM}}^{1-\text{loop}}$ do not depend on the values of the Lorentz vector indices. This only turns out to be the case by means of some non-trivial identities in theta functions.

3.2 Dependence on the point of insertion of the PCOs

Before proceeding, it is convenient to make a quick analysis of the dependence of the world-sheet integrand appearing in (3.2) on the PCO insertion points w_1 and w_2.

Suppose we take the derivative of the amplitude with respect to w_1. We know that the result must be zero because the amplitude should not depend on the point of insertion of the PCO operator. In general this comes about only *after* integrating over the moduli — the differentiation with respect to w_1 gives rise to a total derivative in the integrand. However, in the present case things are more simple. Indeed, substituting $\Pi(w_1) = 2\{Q_{BRST}, \xi(w_1)\}$ in the amplitude and then moving the BRST commutator onto the other operators we find

$$\begin{aligned}
&\partial_{w_1} T^{1-\text{loop}}(\ e^{\pm}\ \to\ e^{\pm}\ +\ \gamma)\ = \\
&-2C_{g=1}\int \mathrm{d}^2\tau \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 \sum_{m_i,n_j} C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \sum_{m^I=\tau,z_1,z_2} \frac{\partial}{\partial m^I} \\
&\langle\langle\ (\bar\eta_{\bar\tau}|\bar b)(\bar\eta_{\bar z_1}|\bar b)(\bar\eta_{\bar z_2}|\bar b)\frac{\partial}{\partial(\eta_{m^I}|b)}\ \{(\eta_\tau|b)(\eta_{z_1}|b)(\eta_{z_2}|b)\}\ |c(z)c(z_1)c(z_2)|^2\ \times \\
&\partial_{w_1}\xi(w_1)\ \Pi(w_2)\ \mathcal{V}^{(-1)}_{\text{photon}}(z,\bar z;k;\epsilon)\ \mathcal{V}^{(-1/2)}(z_1,\bar z_1;k_1;\mathbb{V}_1)\ \mathcal{V}^{(-1/2)}(z_2,\bar z_2;k_2;\mathbb{V}_2)\,\rangle\rangle\ ,
\end{aligned} \tag{3.4}$$

where we used that Π, as well as $\bar c c\mathcal{V}$, are BRST invariant and that [24]

$$\langle\langle\ \{Q_{\text{BRST}}, (\eta_I|b)\}\ldots\rangle\rangle = \langle\langle\ (\eta_I|T_B)\ldots\rangle\rangle = \frac{\partial}{\partial m^I}\langle\langle\ldots\rangle\rangle\ . \tag{3.5}$$

Now, by superghost charge conservation, only the part with superghost number two in $\Pi(w_2)$ can give a non zero contribution to the integrand. But this part of the PCO (the last two terms in eq. (2.13)) is made up only of ghosts and superghosts and thus the only ψ^μ appearing is the one residing in the superghost charge (-1) photon vertex operator (2.56). The Lorentz structure of the total derivative (3.4) is therefore seen to contain only a single gamma-matrix, contracted with the photon polarization ϵ, that is, the total derivative contributes only to the renormalization part of the amplitude, $T_{\text{REN}}^{1-\text{loop}}$.

This is very fortunate, because it means that the *integrands* appearing in the expression for $T_{\text{AMM}}^{1-\text{loop}}$ and $T_{\text{PEDM}}^{1-\text{loop}}$ are *independent* of w_1 and w_2. In particular, the vanishing of

these quantities are not obscured by the presence of any total derivative. Let us also note that $T^{1-\text{loop}}_{\text{REN}}$ is ill-defined on-shell, since the modular integral contains divergencies in the corners of moduli space where the loop is isolated on an external leg, corresponding to the pinching limits $z_1 \to z_2$, $z \to z_1$ and $z \to z_2$, as well as in the limits corresponding to tadpole diagrams ($|z_1 - z| \ll |z_2 - z| \to 0$, $|z_2 - z| \ll |z_1 - z| \to 0$ and $|z_1 - z_2| \ll |z_1 - z| \to 0$). Some regularization and renormalization procedure is needed to properly treat this part of the amplitude. On the other hand, the integrands appearing in $T^{1-\text{loop}}_{\text{AMM}}$ and $T^{1-\text{loop}}_{\text{PEDM}}$ are completely well-behaved in all these pinching limits.

In performing the actual calculation of the amplitude it is convenient to take the limit $w_2 \to z$ so to represent the photon by the zero superghost number vertex operator (2.57). The other PCO we retain at an arbitrary point, $w \equiv w_1$. By taking the limit $w_2 \to z$ in eq. (3.4) it is easy to see that superghost charge conservation now forces the total derivative to vanish altogether, meaning that the integrand must be explicitly independent of $w \equiv w_1$.

One might think that it would be advantageous to take also the limit, say, $w \to z_1$, so as to picture-change one of the "electron/positron" vertex operators; but retaining w at an arbitrary point actually leads to simpler calculations, even though we have to deal with four rather than three vertex insertions. A similar observation was made in ref. [16]. Furthermore, the eventual independence of w provides a powerful check of the result.

The form of the amplitude from which we start is then

$$
\begin{aligned}
&T^{1-\text{loop}}(\ e^{\pm}\ \to\ e^{\pm}\ +\ \gamma)\ = \qquad (3.6)\\
&\quad C_{g=1} \int \mathrm{d}^2\tau \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 \sum_{m_i, n_j} C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\ \langle\langle |(\eta_\tau|b)(\eta_{z_1}|b)(\eta_{z_2}|b)\, c(z)c(z_1)c(z_2)|^2 \times \\
&\quad \Pi(w)\ \mathcal{V}^{(0)}_{\text{photon}}(z,\bar z;k;\epsilon)\ \mathcal{V}^{(-1/2)}(z_1,\bar z_1;k_1;\mathbb{V}_1)\ \mathcal{V}^{(-1/2)}(z_2,\bar z_2;k_2;\mathbb{V}_2)\rangle\rangle\ .
\end{aligned}
$$

3.3 Computation of correlators

If we substitute the explicit form of the vertex operators eqs. (2.57) and (2.58) in eq. (3.6), we obtain after some rearranging of operators

$$
\begin{aligned}
&T^{1-\text{loop}}(\ e^{\pm}\ \to\ e^{\pm}\ +\ \gamma)\ = \qquad (3.7)\\
&\quad -C_{g=1}\frac{\kappa}{\pi}(N_f)^2\ \mathbb{V}_1^A \mathbb{V}_2^B \sum_{m_i,n_j} C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\ e^{i\pi \mathbb{A}\cdot Y\cdot \mathbb{B}} \int \frac{\mathrm{d}^2 k}{\bar k^2 k^2} \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\bar\omega(\bar z)\omega(z)} \prod_{n=1}^{\infty} |1-k^n|^4 \times T_L \times T_R
\end{aligned}
$$

where

$$T_L \; = \; \langle\langle \prod_{l=1}^{7} \prod_{l=15}^{16} \left(\bar{S}^{(\bar{l})}_{\bar{a}_l}(\bar{z}_1) \bar{S}^{(\bar{l})}_{\bar{b}_l}(\bar{z}_2) \right) \; \bar{\partial}\bar{\phi}_{(\overline{17})}(\bar{z}) \bar{S}^{(\overline{17})}_{\bar{a}_{17}}(\bar{z}_1) \bar{S}^{(\overline{17})}_{\bar{b}_{17}}(\bar{z}_2) \,\rangle\rangle \tag{3.8}$$

and

$$\begin{aligned} T_R \; = & \langle\langle \, (\epsilon \cdot \partial X(z) - ik \cdot \psi(z) \, \epsilon \cdot \psi(z)) \; \times \\ & \left(\partial X \cdot \psi(w) + \sum_{m=1}^{2} \psi^m_{(5)} \psi^m_{(6)} \psi^m_{(7)}(w) \right) \prod_{l=0,1,4,5,6,7,9} \left(S^{(l)}_{a_l}(z_1) S^{(l)}_{b_l}(z_2) \right) \; \times \\ & e^{ik\cdot X(z,\bar{z})} e^{ik_1\cdot X(z_1,\bar{z}_1)} e^{ik_2\cdot X(z_2,\bar{z}_2)} e^{\phi(w)} e^{-\frac{1}{2}\phi(z_1)} e^{-\frac{1}{2}\phi(z_2)} \,\rangle\rangle \; . \end{aligned} \tag{3.9}$$

Here we already integrated out the reparametrization ghosts by means of the formula given in appendix B . The multiplier $k = \exp\{2\pi i \tau\}$ should of course not be confused with the photon momentum. Fermion number conservation implies that only the two terms in the supercurrent displayed give rise to a non-zero correlation function.

We now turn our attention to the computation of all correlators appearing in eqs. (3.8) and (3.9). We will not discuss in detail the correlators involving the X^μ fields which are rather trivial and can be easily reconstructed using the Wick theorem, with the contraction given by the bosonic Green function (see Appendix A for conventions).

For each bosonized complex fermion it is convenient to define a correlation function $\langle \ldots \rangle$ where the non-zero mode part of the partition function has been removed

$$\langle\langle \, \mathcal{O}_1(z_1) \ldots \mathcal{O}_N(z_N) \,\rangle\rangle_{(l)} \; = \; \prod_{n=1}^{\infty} (1-k^n)^{-1} \langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_N(z_N) \rangle_{(l)} \; . \tag{3.10}$$

The subscript (l) is there to remind us that the correlator depends on the spin structure. The fundamental correlator $\langle \prod_{i=1}^{N} e^{q_i \phi(z_i)} \rangle$ is given in Appendix A and correlators involving $\partial\phi$ can be obtained from this by differentiation. Notice that

$$\langle \mathbf{1} \rangle_{(l)} = \Theta \begin{bmatrix} \alpha_l \\ \beta_l \end{bmatrix} (0|\tau) \; , \tag{3.11}$$

which vanishes when the spin structure is odd.

The fundamental spin field correlator is

$$\langle S^{(l)}_{a_l}(z_1) S^{(l)}_{b_l}(z_2) \rangle_{(l)} \; = \; \left((\sigma_3^{(l)})^{S_l} \sigma_1^{(l)} \right)_{a_l, b_l} \langle S_+(z_1) S_-(z_2) \rangle_{(l)} \; , \tag{3.12}$$

where we introduced

$$S_l \equiv (1-2\alpha_l)(1+2\beta_l) \; , \tag{3.13}$$

which is 0 (1) mod 2 depending on whether the spin structure $\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}$ is even (odd). Notice that the correlator (3.12) develops a dependence on the sign of the charge a_l whenever the spin structure is odd. The correlator $\langle S_+(z_1)S_-(z_2)\rangle$ is given explicitly in appendix A .

The other correlators that we need are

$$\begin{aligned}
&\langle \bar\partial\bar\phi_{(\overline{17})}(\bar z)\bar S^{(\overline{17})}_{\bar a_{17}}(\bar z_1)\bar S^{(\overline{17})}_{\bar b_{17}}(\bar z_2)\rangle \ = \\
&\qquad \left((\sigma_3^{(\overline{17})})^{1+\overline{S}_{17}}\sigma_1^{(\overline{17})}\right)_{\bar a_{17}\bar b_{17}}\ \langle \bar S_+(\bar z_1)\bar S_-(\bar z_2)\rangle_{(\overline{17})}\ \overline{I}\begin{bmatrix}\bar\alpha_{17}\\ \bar\beta_{17}\end{bmatrix}(\bar z,\bar z_1,\bar z_2)\ , \\
&\langle \psi^\rho(w)\, S^{(0)}_{a_0}(z_1)\, S^{(0)}_{b_0}(z_2)\, S^{(1)}_{a_1}(z_1)\, S^{(1)}_{b_1}(z_2)\rangle \ = \\
&\qquad \frac{1}{\sqrt2}e^{-i\pi a_1 Y_{10} b_0}(\Gamma^\rho(\Gamma^5)^{S_1}\tilde{\mathrm C})_{\alpha\beta}\ \left(\langle S_+(z_1)S_-(z_2)\rangle_{(1)}\right)^2\ \mathcal I\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)\ , \\
&\langle (\sum_{m=1}^{2}\psi^m_{(5)}\psi^m_{(6)}\psi^m_{(7)})(w)\, S^{(5)}_{a_5}(z_1)\, S^{(5)}_{b_5}(z_2)\, S^{(6)}_{a_6}(z_1)\, S^{(6)}_{b_6}(z_2)\, S^{(7)}_{a_7}(z_1)\, S^{(7)}_{b_7}(z_2)\rangle = \\
&\qquad -\frac{1}{2\sqrt2}\left(\tilde{\mathbf M}\,((\sigma_3^{(5)})^{S_5}\sigma_1^{(5)}\otimes(\sigma_3^{(6)})^{S_6}\sigma_1^{(6)}\otimes(\sigma_3^{(7)})^{S_7}\sigma_1^{(7)})\right)_{a_5a_6a_7,b_5b_6b_7}\ \times \\
&\qquad \prod_{l=5,6,7}\langle S_+(z_1)S_-(z_2)\rangle_{(l)}\ \prod_{l=5,6,7}\mathcal I\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(w,z_1,z_2)\ , \\
&\langle \psi^\mu\psi^\nu(z)\,\psi^\rho(w)\, S^{(0)}_{a_0}(z_1)\, S^{(0)}_{b_0}(z_2)\, S^{(1)}_{a_1}(z_1)\, S^{(1)}_{b_1}(z_2)\rangle = \ -\frac{1}{\sqrt2}e^{-i\pi a_1 Y_{10} b_0}\ \times \\
&\qquad\quad \Big\{(\Gamma^{\mu\nu\rho}(\Gamma_5)^{S_1}\tilde{\mathrm C})_{\alpha\beta}\,G^-\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(z,w;z_1,z_2)\ + \\
&\qquad\qquad \left((g^{\mu\rho}\Gamma^\nu-g^{\nu\rho}\Gamma^\mu)(\Gamma_5)^{S_1}\tilde{\mathrm C}\right)_{\alpha\beta}G^+\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(z,w;z_1,z_2)\Big\}\ \times \\
&\qquad\quad \left(\langle S_+(z_1)S_-(z_2)\rangle_{(1)}\right)^2\ \mathcal I\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(z,z_1,z_2)\ , \\
&\langle \psi^\mu\psi^\nu(z)\, S^{(0)}_{a_0}(z_1)\, S^{(0)}_{b_0}(z_2)\, S^{(1)}_{a_1}(z_1)\, S^{(1)}_{b_1}(z_2)\rangle = \ -\frac12 e^{-i\pi a_1 Y_{10} b_0}\ \times \\
&\qquad\quad \left(\Gamma^{\mu\nu}(\Gamma^5)^{S_1}\tilde{\mathrm C}\right)_{\alpha\beta}\ \left(\langle S_+(z_1)S_-(z_2)\rangle_{(1)}\right)^2\ I\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(z,z_1,z_2)\ .
\end{aligned}\tag{3.14}$$

Here $\Gamma^{\mu\nu}$ and $\Gamma^{\mu\nu\rho}$ are products of gamma matrices antisymmetrized with unit weight; we also introduced the abbreviations

$$\begin{aligned}
\tilde{\mathrm C}_{\alpha\beta} &\equiv \delta_{a_0+b_0}\delta_{a_1+b_1}e^{i\pi a_1 Y_{10} b_0}\\
\tilde{\mathbf M} &\equiv \sigma_1^{(5)}\sigma_1^{(6)}\sigma_1^{(7)}-\sigma_2^{(5)}\sigma_2^{(6)}\sigma_2^{(7)}
\end{aligned}\tag{3.15}$$

and defined the following functions of the world-sheet coordinates

$$\mathcal I\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,z_1,z_2) = \sqrt{\frac{E(z_1,z_2)}{E(z,z_1)E(z,z_2)}}\,\frac{\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\mu_z|\tau)}{\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\frac12\nu_{12}|\tau)}\ , \tag{3.16}$$

$$
\begin{aligned}
I\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2) &= \partial_z \log \frac{E(z,z_1)}{E(z,z_2)} + 2\frac{\omega(z)}{2\pi i}\partial_\nu \log\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} \\
&= \left(\mathcal{I}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2)\right)^2 \ , \\
G^+\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,w;z_1,z_2) &= \frac{1}{2E(z,w)}\left\{\frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\rho_{z,w}|\tau)}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\frac{1}{2}\nu_{12}|\tau)}\sqrt{\frac{E(z,z_1)E(w,z_2)}{E(w,z_1)E(z,z_2)}} + \right. \\
&\qquad \left. \frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\rho_{w,z}|\tau)}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\frac{1}{2}\nu_{12}|\tau)}\sqrt{\frac{E(w,z_1)E(z,z_2)}{E(z,z_1)E(w,z_2)}}\right\} \\
&= \frac{\mathcal{I}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](w,z_1,z_2)}{\mathcal{I}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2)}\left\{\partial_z \log\frac{E(z,w)}{\sqrt{E(z,z_1)E(z,z_2)}} + \right. \\
&\qquad \left. \frac{\omega(z)}{2\pi i}\partial_\nu \log\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\nu|\tau)|_{\nu=\mu_w}\right\} \ , \\
G^-\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,w;z_1,z_2) &= \frac{1}{2E(z,w)}\left\{\frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\rho_{z,w}|\tau)}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\frac{1}{2}\nu_{12}|\tau)}\sqrt{\frac{E(z,z_1)E(w,z_2)}{E(w,z_1)E(z,z_2)}} - \right. \\
&\qquad \left. \frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\rho_{w,z}|\tau)}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\frac{1}{2}\nu_{12}|\tau)}\sqrt{\frac{E(w,z_1)E(z,z_2)}{E(z,z_1)E(w,z_2)}}\right\} \\
&= \frac{1}{2}\mathcal{I}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2)\mathcal{I}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](w,z_1,z_2) \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_{12} &\equiv \int_{z_2}^{z_1} \tfrac{\omega}{2\pi i} \\
\mu_z &\equiv \int^{z} \tfrac{\omega}{2\pi i} - \tfrac{1}{2}\int^{z_1} \tfrac{\omega}{2\pi i} - \tfrac{1}{2}\int^{z_2} \tfrac{\omega}{2\pi i} \\
\rho_{z,w} &\equiv \int_{w}^{z} \tfrac{\omega}{2\pi i} + \tfrac{1}{2}\int_{z_2}^{z_1} \tfrac{\omega}{2\pi i} \ .
\end{aligned}
\tag{3.17}
$$

To arrive at the correlators (3.14) is quite tedious. In the next subsection we give an explicit example. Notice that in (3.16) we give two different expressions for the functions $I\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ and $G^\pm\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$. These two expressions appear when we compute the correlators (3.14) for different values of the Lorentz vector indices. Lorentz covariance implies that the two expressions are identical. This may also be proved directly. In appendix E we sketch the proof of the equivalence of the two forms given for the function $I\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$.

Finally, the correlator involving the superghosts is computed in Appendix B and is given by

$$\langle\langle\, e^{\phi(w)} e^{-\frac{1}{2}\phi(z_1)} e^{-\frac{1}{2}\phi(z_2)}\, \rangle\rangle \;=\; \tag{3.18}$$
$$(-1)^{S_1} k^{1/2} \prod_{n=1}^{\infty} (1-k^n) \frac{(\omega(z_1)\omega(z_2))^{1/2}}{\omega(w)} \frac{1}{\langle S_+(z_1) S_-(z_2)\rangle_{(0)} \mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)} \;.$$

3.4 The explicit computation of a correlator

In this subsection we outline the computation of the correlator

$$\langle \psi^\mu \psi^\nu(w) S^{(0)}_{a_0}(z_1) S^{(0)}_{b_0}(z_2) S^{(1)}_{a_1}(z_1) S^{(1)}_{b_1}(z_2)\rangle \;. \tag{3.19}$$

The other correlators in (3.14) can be obtained in a similar way. We will compute (3.19) in two cases: For $\mu=0$, $\nu=1$ and for $\mu=0$, $\nu=2$. The other cases can be worked out similarly.

Let us consider first $\mu=0$, $\nu=1$. Since $\psi^0\psi^1(w) = i\partial\phi_{(0)}(w)$ we get

$$\langle \psi^0 \psi^1(w) S^{(0)}_{a_0}(z_1) S^{(0)}_{b_0}(z_2) S^{(1)}_{a_1}(z_1) S^{(1)}_{b_1}(z_2)\rangle \;=\; \tag{3.20}$$
$$\langle i\partial\phi(w) S_{a_0}(z_1) S_{b_0}(z_2)\rangle_{(0)} \, \langle S_{a_1}(z_1) S_{b_1}(z_2)\rangle_{(1)} \;.$$

Bosonizing the spin fields and using the formulæ given in Appendix A , we get

$$\langle \psi^0 \psi^1(w) S^{(0)}_{a_0}(z_1) S^{(0)}_{b_0}(z_2) S^{(1)}_{a_1}(z_1) S^{(1)}_{b_1}(z_2)\rangle \;=\; i a_0 \left((\sigma_3)^{S_0}\sigma_1\right)_{a_0 b_0} \left((\sigma_3)^{S_1}\sigma_1\right)_{a_1 b_1} \times$$
$$\left(\partial_w \log\frac{E(w,z_1)}{E(w,z_2)} + 2\frac{\omega(w)}{2\pi i}\partial_\nu \log\Theta\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix}(\nu|\tau)\big|_{\nu=\frac{1}{2}\nu_{12}}\right) \times$$
$$\langle S_+(z_1) S_-(z_2)\rangle_{(0)} \langle S_+(z_1) S_-(z_2)\rangle_{(1)} \;. \tag{3.21}$$

Since by the definitions (3.15) and (2.53)

$$i a_0 \left((\sigma_3)^{S_0}\sigma_1\right)_{a_0 b_0} \left((\sigma_3)^{S_1}\sigma_1\right)_{a_1 b_1} = -\tfrac{1}{2}\left(\Gamma^{01}\left(\Gamma^5\right)^{S_1}\tilde{\mathrm{C}}\right)_{\alpha\beta} e^{-i\pi a_1 Y_{10} b_0} \;, \tag{3.22}$$

eq. (3.20) yields the result quoted in (3.14), with $I\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ given by the first expression appearing in (3.16).

We now consider the case $\mu=0$, $\nu=2$, where we have

$$\psi^0\psi^2(w) = \tfrac{1}{2}\left(e^{\phi^{(0)}(w)} c_{(0)} + e^{-\phi^{(0)}(w)} (c_{(0)})^{-1}\right)\left(e^{\phi^{(1)}(w)} c_{(1)} + e^{-\phi^{(1)}(w)} (c_{(1)})^{-1}\right) \;, \tag{3.23}$$

and then

$$
\begin{aligned}
\langle\psi^0\psi^2(w)S^{(0)}_{a_0}(z_1)S^{(0)}_{b_0}(z_2)S^{(1)}_{a_1}(z_1)S^{(1)}_{b_1}(z_2)\rangle \; = & \\
-\tfrac{1}{2}\left(\delta_{1+a_0+b_0}+(-1)^{S_0}\delta_{-1+a_0+b_0}\right)\left(\delta_{1+a_1+b_1}+(-1)^{S_1}\delta_{-1+a_1+b_1}\right) \; \times & \\
\frac{E(z_1,z_2)}{E(w,z_1)E(w,z_2)}\left(\frac{\Theta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\mu_w|\tau)}{\Theta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\frac{1}{2}\nu_{12}|\tau)}\right)^2 \; \times & \\
\langle S_+(z_1)S_-(z_2)\rangle_{(0)}\langle S_+(z_1)S_-(z_2)\rangle_{(1)} \; . &
\end{aligned}
\tag{3.24}
$$

It is straightforward to check that

$$
\begin{aligned}
\left(\delta_{1+a_0+b_0}+(-1)^{S_0}\delta_{-1+a_0+b_0}\right)\left(\delta_{1+a_1+b_1}+(-1)^{S_1}\delta_{-1+a_1+b_1}\right) \; = & \\
\left(\Gamma^{02}\left(\Gamma^5\right)^{S_1}\tilde{\mathrm{C}}\right)_{\alpha\beta} e^{-i\pi a_1 Y_{10} b_0} \; , &
\end{aligned}
\tag{3.25}
$$

so that we obtain again the result quoted in (3.14) with $I\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ now given by the second expression in (3.16). Thus, once the equality of the two expressions for $I\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is proven (see Appendix E), one obtains a Lorentz covariant formula for the correlator (3.19):

$$
\begin{aligned}
\langle\psi^\mu\psi^\nu(w)S^{(0)}_{a_0}(z_1)S^{(0)}_{b_0}(z_2)S^{(1)}_{a_1}(z_1)S^{(1)}_{b_1}(z_2)\rangle \; = & \\
-\frac{1}{2}e^{-i\pi a_1 Y_{10} b_0}\left(\Gamma_{\mu\nu}(\Gamma^5)^{S_1}\tilde{\mathrm{C}}\right)_{\alpha\beta} I\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)\left(\langle S_+(z_1)S_-(z_2)\rangle_{(1)}\right)^2 . &
\end{aligned}
\tag{3.26}
$$

In the same way one derives all the formulæ in eq. (3.14).

3.5 Using the GSO projections and Dirac equation

To arrive at the final form of the amplitude, we have yet to make various simplifications. Due to lack of space and the obvious unnecessity of entering into too many details, we will just indicate the main steps.

First of all, notice that substituting the correlators (3.14) into eq. (3.7), we do not reconstruct directly the charge conjugation matrix $\mathbb{C}$ or the mass matrix $\mathbf{M}$. Indeed, even if in eq. (3.7) there is an overall phase factor $e^{i\pi\mathbb{A}\cdot Y\cdot\mathbb{B}}$ there also appears a factor of the form $\left(\sigma^{(l)}_n(\sigma^{(l)}_3)^{S_l}\sigma_1\right)_{a_l,b_l} = \left(\sigma^{(l)}_n(\sigma^{(l)}_3)^{S_l}\right)_{a_l,b'_l}(\sigma_1)_{b'_l,b_l}$ $(n=0,1,2,3)$ for each left and right moving fermion with R boundary conditions. What we need to do is to rewrite $e^{i\pi\mathbb{A}\cdot Y\cdot\mathbb{B}} = e^{i\pi\mathbb{B}'\cdot Y\cdot\mathbb{B}}e^{i\pi(\mathbb{A}-\mathbb{B}')\cdot Y\cdot\mathbb{B}}$ where the first factor is what we need to reconstruct the

$\mathbb{C}$ matrix and the second can be rewritten as a product of σ_3 matrices acting directly on the spinor $\mathbb{V}_2$. In this way one obtains the following relations

$$
\begin{aligned}
& e^{i\pi \mathbb{A}\cdot Y \cdot \mathbb{B}} \prod_{l=1}^{7}\prod_{l=15}^{16} \left((\sigma_3^{(\bar{l})})^{\bar{S}_l}\sigma_1^{(\bar{l})}\right)_{\bar{a}_l\bar{b}_l} \left((\sigma_3^{(\overline{17})})^{1+\bar{S}_{17}}\sigma_1^{(\overline{17})}\right)_{\bar{a}_{17}\bar{b}_{17}} \times \\
& \qquad \left(\Gamma_* \left(\Gamma^5\right)^{S_1} \tilde{\mathrm{C}}\right)_{\alpha\beta} \prod_{l=4,5,6,7,9} \left((\sigma_3^{(l)})^{S_l}\sigma_1^{(l)}\right)_{a_l b_l} e^{-i\pi a_1 Y_{10} b_0} = \\
& - \left(\left(\Gamma_{SO(14)}\right)^{1+\bar{S}_1} \left(\Gamma_{SO(4)}\right)^{1+\bar{S}_{15}} \left(\sigma_3^{(\overline{17})}\right)^{\bar{S}_{17}} \Gamma_* \mathbf{\Gamma_S} \mathbb{C}\right)_{AB} ,
\end{aligned} \tag{3.27}
$$

where Γ_* denotes either Γ^ρ or $\Gamma^{\mu\nu\rho}$; and

$$
\begin{aligned}
& e^{i\pi \mathbb{A}\cdot Y \cdot \mathbb{B}} \prod_{l=1}^{7}\prod_{l=15}^{16} \left((\sigma_3^{(\bar{l})})^{\bar{S}_l}\sigma_1^{(\bar{l})}\right)_{\bar{a}_l\bar{b}_l} \left((\sigma_3^{(\overline{17})})^{1+\bar{S}_{17}}\sigma_1^{(\overline{17})}\right)_{\bar{a}_{17}\bar{b}_{17}} \left(\Gamma_* \left(\Gamma^5\right)^{S_1} \tilde{\mathrm{C}}\right)_{\alpha\beta} \times \\
& \qquad \left(\tilde{\mathbf{M}} \prod_{l=4,5,6,7,9} \left((\sigma_3^{(l)})^{S_l}\sigma_1^{(l)}\right)_{a_4a_5a_6a_7a_9, b_4b_5b_6b_7b_9} e^{-i\pi a_1 Y_{10} b_0}\right) = \\
& - 2i \left(\left(\Gamma_{SO(14)}\right)^{1+\bar{S}_1} \left(\Gamma_{SO(4)}\right)^{1+\bar{S}_{15}} \left(\sigma_3^{(\overline{17})}\right)^{\bar{S}_{17}} \Gamma_* \mathbf{M} \mathbf{\Gamma_S} \mathbb{C}\right)_{AB} ,
\end{aligned} \tag{3.28}
$$

with Γ_* now denoting either 1 or $\Gamma^{\mu\nu}$. Here we defined

$$
\mathbf{\Gamma_S} \equiv \left(\Gamma^5\right)^{S_1} \otimes_{l=4,5,6,7,9} \left(\sigma_3^{(l)}\right)^{S_l} . \tag{3.29}
$$

In equations (3.27) and (3.28) we used the cocycles' choice (2.47) since the general expression turns out to be quite long and in this subsection we are interested only in indicating to the reader the various steps needed to arrive at the final form of the amplitude. In any case the reader can easily obtain the corresponding expressions depending explicitly on the cocycles. The form of these equations, when one leaves unspecified the choice of cocycles, differs from equations (3.27) and (3.28) only for some signs appearing in the definition of the gamma matrices and of the mass matrix $\mathbf{M}$. As it will be immediately obvious, the fact that in eqs. (3.27) and (3.28) we used the cocycles' choice (2.47) has no consequences on the generality of what follows.

Each term in our amplitude now has the following general structure for what concerns the dependence on the external gauge and Lorentz spinor indices:

$$
(\Gamma_{SO(14)})^{1+\bar{S}_1} (\Gamma_{SO(4)})^{1+\bar{S}_{15}} \left((\sigma_3^{(\overline{17})})^{\bar{S}_{17}}\right) \otimes \mathcal{O}\, \mathbf{\Gamma_S} \mathbb{C} , \tag{3.30}
$$

where $\mathcal{O}$ denotes either Γ^ρ, $\Gamma^{\mu\nu\rho}$, $\mathbf{M}$ or $\Gamma^{\mu\nu}\mathbf{M}$. Remembering that these structures are sandwiched between the '1' and '2' spinors, we can use the GSO projection conditions (1.41) to rewrite (3.30) on the form

$$\Gamma_{SO(14)}\Gamma_{SO(4)}(\sigma_3^{(17)})^{\bar{S}_{17}+S_4+S_6+S_7} \otimes \mathcal{O}\left(\Gamma^5\right)^{S_1+S_5+S_6+S_7} \mathbb{C}\ \exp\{2\pi i[K_{GSO}]\}\ , \tag{3.31}$$

where

$$\begin{aligned} K_{GSO} &= (k_{00}+k_{01}+k_{03}+k_{13})S_5 + \\ &\quad (\tfrac{1}{2}+k_{02}+k_{12}+k_{13}+k_{23}+k_{04}+k_{14}+k_{34})S_4 + \\ &\quad (k_{00}+k_{01}+k_{03}+k_{04}+k_{14}+k_{34})S_7 + \\ &\quad (k_{00}+k_{01}+k_{02}+k_{03}+k_{12}+k_{23})S_6\ , \end{aligned} \tag{3.32}$$

and we used the fact (following directly from the form of the $\mathbf{W}$-vectors (1.24)) that $\bar{S}_1 = S_7$ and $\bar{S}_{15} = S_6 = S_9$. It is straightforward to verify that

$$\bar{S}_{17} + S_4 \overset{\text{MOD } 2}{=} S_1 + S_5 + \bar{S}_{18} + S_2 + S_3 + S_8 \tag{3.33}$$

and, since the amplitude contains the overall factor

$$\prod_{l=8}^{14} \overline{\Theta}\begin{bmatrix}\bar{\alpha}_l\\ \bar{\beta}_l\end{bmatrix}(0|\bar{\tau}) \prod_{l=18}^{22} \overline{\Theta}\begin{bmatrix}\bar{\alpha}_l\\ \bar{\beta}_l\end{bmatrix}(0|\bar{\tau}) \prod_{l=2,3,8,10} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(0|\tau)\ , \tag{3.34}$$

which vanishes whenever $\bar{S}_8$, $\bar{S}_{18}$, S_2, S_3, S_8 or S_{10} equals 1 (mod 2), it is legitimate to use the following identity

$$\bar{S}_{17} + S_4 \overset{\text{MOD } 2}{=} S_1 + S_5\ . \tag{3.35}$$

Then it is convenient to introduce

$$S \equiv S_1 + S_5 + S_6 + S_7 \overset{\text{MOD } 2}{=} n_2 m_4 + n_4 m_2 + \bar{S}_{18} + S_2 + S_3 + S_8 \tag{3.36}$$

so that the general term (3.30) becomes

$$\Gamma_{SO(14)}\Gamma_{SO(4)}\left(\sigma_3^{(17)}\right)^S \otimes \mathcal{O}\left(\Gamma^5\right)^S \mathbb{C}\ \exp\{2\pi i[K_{GSO}]\}\ . \tag{3.37}$$

The phase factor $\exp\{2\pi i K_{GSO}\}$ combines with the summation coefficient $C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}$, given by eq. (1.17), which in the particular case of our toy model can be written as

$$\begin{aligned} C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \; = \; & \exp\{2\pi i[\tfrac{1}{2}(m_0+n_0)+\tfrac{1}{2}n_0(m_2+m_3+m_4)+\tfrac{1}{2}n_1'(m_2+m_3+m_4)+ \\ & \tfrac{1}{2}m_3n_3+\tfrac{1}{2}m_4n_4+\tfrac{1}{2}m_3n_4+S_4(k_{13}+k_{23}+k_{34}+\tfrac{1}{2})+ \\ & S_1(k_{00}+k_{01}+k_{02}+k_{03}+k_{04}+k_{12}+k_{23}+k_{24}+\tfrac{1}{2})+ \\ & S_5(k_{02}+k_{04}+k_{12}+k_{13}+k_{23}+k_{24})+ \\ & S_6(k_{04}+k_{24})+S_7(k_{04}+k_{23}+k_{24}+k_{34})+ \\ & (k_{04}+k_{14}-k_{02}-k_{12})(m_4n_0+n_4m_0+m_1'n_4+m_4n_1')]\}\ , \end{aligned} \tag{3.38}$$

where we put $\bar{S}_8, \bar{S}_{18}, S_2, S_3, S_8$ and S_{10} equal to zero throughout, made use of the identity

$$S_1+\bar{S}_{18}+S_4+S_8+m_1(n_2+n_3+n_4)+n_1(m_2+m_3+m_4) \overset{\text{MOD } 2}{=} 0 \tag{3.39}$$

as well as (3.35) and introduced $m_1' = m_0+m_1$ and $n_1' = n_0+n_1$ for future convenience.

Putting together all the phases (including the $(-1)^{S_1}$ which comes from the superghost correlator (3.18)) we arrive at the overall phase

$$\begin{aligned} K^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \; & \equiv (-1)^{S_1} e^{2\pi i K_{GSO}} C^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \; = \\ & = \; \exp\left\{2\pi i\left[\tfrac{1}{2}(m_0+n_0)+\tfrac{1}{2}n_0(m_2+m_3+m_4)+\right.\right. \\ & \quad \tfrac{1}{2}n_1'(m_2+m_3+m_4)+\tfrac{1}{2}m_3n_3+\tfrac{1}{2}m_4n_4+\tfrac{1}{2}m_3n_4+ \\ & \quad (k_{00}+k_{01}+k_{02}+k_{03}+k_{04}+k_{12}+k_{23}+k_{24})\ S+ \\ & \quad \left.\left.(k_{04}+k_{14}-k_{02}-k_{12})(S_4+S_7+m_4n_0+m_0n_4+m_1'n_4+m_4n_1')\right]\right\}\ . \end{aligned} \tag{3.40}$$

It is obvious from eq. (1.37) that for spacetime supersymmetric models, the last term in $K^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}$ vanishes. This will be the key point in the proof of the vanishing of the Anomalous Magnetic Moment for spacetime supersymmetric models.

Finally, to rewrite all Lorentz structures appearing in the amplitude on the form of eq. (3.3) one has to use the on-shell conditions $k^2=0$, $k_1^2=k_2^2=-\frac{1}{2}$, momentum conservation $k+k_1+k_2=0$, and the Dirac equations

$$\mathbf{V}_1^T(\not{k}_1+\mathbf{M}) = 0 = (\not{k}_2-\mathbf{M})\mathbf{C}\mathbf{V}_2 \tag{3.41}$$

from which one may derive the following useful identities

$$\begin{aligned} & \mathbf{V}_1^T(\Gamma^5)^S\not{k}\mathbf{C}\mathbf{V}_2\epsilon\cdot k_1 = \tfrac{1}{2}(1-(-1)^S)\epsilon_\mu k_\nu \mathbf{V}_1^T\mathbf{M}\Gamma^5\Gamma^{\mu\nu}\mathbf{C}\mathbf{V}_2 \\ & \mathbf{V}_1^T\tfrac{1}{2}(\not{k}_1-\not{k}_2)(\Gamma^5)^S\mathbf{C}\mathbf{V}_2\epsilon\cdot k_1 = \tfrac{1}{2}(1+(-1)^S)\left\{-\mathbf{V}_1^T\mathbf{M}^2\not{\epsilon}\mathbf{C}\mathbf{V}_2+\tfrac{1}{2}\mathbf{V}_1^T\mathbf{M}\Gamma^{\mu\nu}\mathbf{C}\mathbf{V}_2\epsilon_\mu k_\nu\right\}\ . \end{aligned} \tag{3.42}$$

3.6 The final form of the amplitude

In this subsection we display the final form of the amplitude that we arrive at following the steps described in the previous subsections. It is equivalent to the form of a field theory amplitude where the internal momenta are already integrated away whereas the integrals over the Schwinger proper-times are still to be done. In other words, all Lorentz algebra is already done and what is left to do is an adimensional integral.

We present the partial amplitudes $T^{1-\text{loop}}_{\text{REN,AMM,PEDM}}$ as defined in eq. (3.3), as follows

$$\begin{aligned} T^{1-\text{loop}}_{\text{REN,AMM,PEDM}} = C_{g=1}\frac{\kappa}{\pi}(N_f)^2 i e^{i\pi\varphi_c}\frac{1}{4\sqrt{2}} & \left(\bar{V}^T_{SO(14),1}\Gamma_{SO(14)}\mathrm{C}_{SO(14)}\bar{V}_{SO(14),2}\right) \times \\ & \left(\bar{V}^T_{SO(4),1}\Gamma_{SO(4)}\mathrm{C}_{SO(4)}\bar{V}_{SO(4),2}\right) \left(\bar{V}^T_{U(1),1}(\sigma_3^{(\overline{17})})^S\sigma_1^{(\overline{17})}\bar{V}_{U(1),2}\right) \times \\ & \sum_{n_i,m_i} K^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \int \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\tau)^2}\frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\bar{\omega}(\bar{z})\omega(z)} \exp\left\{2\pi i(\bar{\tau}-\tfrac{1}{2}\tau)\right\} \times \\ & \exp\left\{\tfrac{1}{2}G_B(z_1,\bar{z}_1,z_2,\bar{z}_2)\right\} \times \bar{\mathcal{Z}}_L \times \mathcal{Z}_R \times \mathcal{I}_{\text{REN,AMM,PEDM}} \end{aligned} \tag{3.43}$$

with

$$\begin{aligned} \bar{\mathcal{Z}}_L =& (\bar{\eta}(\bar{\tau}))^{-24}\prod_{l=1}^{7}\prod_{l=15}^{17}\bar{\Theta}\begin{bmatrix}\bar{\alpha}_l\\ \bar{\beta}_l\end{bmatrix}(\tfrac{1}{2}\bar{\nu}_{12}|\bar{\tau}) \times \\ & \prod_{l=8}^{14}\prod_{l=18}^{22}\bar{\Theta}\begin{bmatrix}\bar{\alpha}_l\\ \bar{\beta}_l\end{bmatrix}(0|\bar{\tau}) \times (\bar{E}(\bar{z}_1,\bar{z}_2))^{-5/2}\bar{I}\begin{bmatrix}\bar{\alpha}_{17}\\ \bar{\beta}_{17}\end{bmatrix}(\bar{z},\bar{z}_1,\bar{z}_2) , \end{aligned} \tag{3.44}$$

$$\begin{aligned} \mathcal{Z}_R =& (\eta(\tau))^{-12}\frac{\sqrt{\omega(z_1)\omega(z_2)}}{\omega(w)}\prod_{l=1,4,5,6,7,9}\Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(\tfrac{1}{2}\nu_{12}|\tau) \times \\ & \prod_{l=2,3,8,10}\Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(0|\tau) \times (E(z_1,z_2))^{-3/2} , \end{aligned} \tag{3.45}$$

and finally

$$\begin{aligned} \mathcal{I}_{\text{REN}} =& (1+(-1)^S)\left\{\partial_z G_B(z,z_1) - \partial_z G_B(z,z_2)\right\} \times \\ & \left\{\partial_w G_B(w,z_1) - \partial_w G_B(w,z_2) - \frac{\prod_{l=5,6,7}\mathcal{I}\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(w,z_1,z_2)}{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)}\right\} , \end{aligned} \tag{3.46}$$

$$
\begin{aligned}
\mathcal{I}_{\mathrm{AMM}} = & -\tfrac{1}{2}(1+(-1)^S)\times \\
& \left\{\partial_z G_B(z,z_1) - \partial_z G_B(z,z_2) - I\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(z,z_1,z_2)\right\}\times \\
& \left\{\partial_w G_B(w,z_1) - \partial_w G_B(w,z_2) - \frac{\prod_{l=5,6,7}\mathcal{I}\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(w,z_1,z_2)}{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)}\right\}\ ,
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
\mathcal{I}_{\mathrm{PEDM}} = & (1-(-1)^S)\left\{(\partial_z G_B(z,z_1) - \partial_z G_B(z,z_2))\times\right. \\
& \Big(\partial_w G_B(w,z) - \tfrac{1}{2}\partial_w G_B(w,z_1) - \tfrac{1}{2}\partial_w G_B(w,z_2) + \\
& \left.\tfrac{1}{2}\frac{\prod_{l=5,6,7}\mathcal{I}\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(w,z_1,z_2)}{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)}\right) + \\
& (\partial_w G_B(w,z_1) - \partial_w G_B(w,z_2))\, G^+\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(z,w;z_1,z_2)\frac{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(z,z_1,z_2)}{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)} - \\
& \left.\tfrac{1}{2} I\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(z,z_1,z_2)\frac{\prod_{l=5,6,7}\mathcal{I}\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(w,z_1,z_2)}{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)}\right\}\ .
\end{aligned}
\tag{3.48}
$$

Notice that spin structures with $S = 0 \bmod 2$ contribute to $T^{1-\mathrm{loop}}_{\mathrm{REN}}$ and $T^{1-\mathrm{loop}}_{\mathrm{AMM}}$ and those with $S = 1 \bmod 2$ contribute to $T^{1-\mathrm{loop}}_{\mathrm{PEDM}}$.

3.7 w independence

As a first check on the correctness of our computation, we verify that eq. (3.43) is independent of w, the point of insertion of the PCO operator.

For $\mathcal{I}_{\mathrm{REN}}$ and $\mathcal{I}_{\mathrm{AMM}}$ this is simple enough. One may verify that for all spin structures contributing to the REN and AMM partial amplitudes (i.e. for which $S = 0 \bmod 2$ and the factor (3.34) is nonzero) it is possible to write

$$
\frac{\prod_{l=5,6,7}\mathcal{I}\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(w,z_1,z_2)}{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)} = I\begin{bmatrix}\alpha_L\\ \beta_L\end{bmatrix}(w,z_1,z_2)\ , \tag{3.49}
$$

where $L = 6$ if $(m_4 = m_2, n_4 = n_2)$, $L = 5$ otherwise. Thus the term which depends on w in $T^{1-\mathrm{loop}}_{\mathrm{REN}}$ and $T^{1-\mathrm{loop}}_{\mathrm{AMM}}$ becomes

$$
\frac{1}{\omega(w)}\left\{\partial_w G_B(w,z_1) - \partial_w G_B(w,z_2) - I\begin{bmatrix}\alpha_L\\ \beta_L\end{bmatrix}(w,z_1,z_2)\right\}\ = \tag{3.50}
$$

$$\frac{1}{\omega(w)}\left\{\frac{\omega(w)}{\log|k|}\log\left|\frac{z_2}{z_1}\right| - 2\frac{\omega(w)}{2\pi i}\partial_\nu \log\Theta\begin{bmatrix}\alpha_L\\ \beta_L\end{bmatrix}(\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}}\right\} ,$$

which is explicitly independent of w.

For the PEDM things are more complicated. We have been able to prove the w independence only implicitly, i.e. we have checked that the quantity $\frac{1}{\omega(w)}\mathcal{I}_{\rm PEDM}(w)$, which is a meromorphic function of w on the torus, does not have zeros and that the residues at all poles vanish. Thus it is a constant (as a function of w) and hence independent of w. We do not reproduce the details of this proof here since we will not need it in what follows.

4. The Vanishing of the AMM and PEDM

We will now briefly discuss the properties of the $T^{1-\rm loop}_{\rm AMM}$ and $T^{1-\rm loop}_{\rm PEDM}$. We will not discuss the $T^{1-\rm loop}_{\rm REN}$ term since this requires a regularization and renormalization procedure, as pointed out in the introduction and in section §3.2.

4.1 The vanishing of the PEDM

Let us consider first the PEDM. Since only spin structures with $S = 1 \bmod 2$ contribute to $T^{1-\rm loop}_{\rm PEDM}$, it is clear from eq. (3.43) that this part of the amplitude has an anomalous dependence on the sign of the $U(1)$ charge, as compared to the tree-level amplitude and the AMM-part of the 1-loop amplitude, because of the presence of an extra factor $(\sigma_3^{(17)})$. [3] Hence, if we normalize our spinors $\mathbb{V}_1$ and $\mathbb{V}_2$ (see appendix D for details) such that

$$T^{\rm tree}(\ e^+\ \to\ e^+\ +\ \gamma) = -T^{\rm tree}(\ e^-\ \to\ e^-\ +\ \gamma)\ , \tag{4.1}$$

which is the behaviour consistent with charge conjugation invariance of the S-matrix (since the photon is a charge conjugation eigenstate with eigenvalue -1), it is clear that we will then find

$$T^{1-\rm loop}_{\rm PEDM}(\ e^+\ \to\ e^+\ +\ \gamma) = +T^{1-\rm loop}_{\rm PEDM}(\ e^-\ \to\ e^-\ +\ \gamma)\ , \tag{4.2}$$

i.e. the PEDM, besides violating P and T, also violates C, and then CPT. But as it was pointed out in the introduction, the KLT models should not violate CPT perturbatively (see refs. [20,21,22]). Then $T^{1-\rm loop}_{\rm PEDM}$ must vanish.

[3] We remind the reader that the $U(1)$ factor of the gauge group is associated with the $\overline{17}$ world-sheet fermion.

It is possible to show that for spacetime supersymmetric models, the sum over the spin structures implementing the spacetime supersymmetry makes $T^{1-\text{loop}}_{\text{PEDM}}$ vanish point by point in moduli space. We will not show this in details, but it works in a similar, although more complicated, manner as for the AMM which we will consider in the next subsection. Anyway, for the non spacetime supersymmetric models the sum over the spin structures does not vanish point by point in moduli space. [4]

Indeed the reason for the vanishing of $T^{1-\text{loop}}_{\text{PEDM}}$ is more general. One can show that the following identity holds:

$$\frac{\prod_{l=5,6,7} \mathcal{I}\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(w,z_1,z_2)}{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_1,z_2)} \;=\; (-1)^{1+S}\,\frac{\prod_{l=5,6,7} \mathcal{I}\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(w,z_2,z_1)}{\mathcal{I}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(w,z_2,z_1)}\;. \tag{4.3}$$

Using this identity it is possible to prove that

$$\begin{aligned} T_{\text{REN}}(z_1,z_2) &= +\,T_{\text{REN}}(z_2,z_1) \\ T_{\text{AMM}}(z_1,z_2) &= +\,T_{\text{AMM}}(z_2,z_1) \\ T_{\text{PEDM}}(z_1,z_2) &= -\,T_{\text{PEDM}}(z_2,z_1)\;, \end{aligned} \tag{4.4}$$

where $T_{\text{REN,AMM,PEDM}}(z_1,z_2)$ are the same expressions as for $T^{1-\text{loop}}_{\text{REN,AMM,PEDM}}$ in eq. (3.43) but stripped of the integrals $\int d^2z_1 d^2z_2$. Thus for the PEDM the integrand is odd under the exchange of $z_1 \leftrightarrow z_2$, and then it vanishes.

4.2 The vanishing of the AMM for SUSY models

Finally, we show that the AMM vanishes for spacetime supersymmetric models according to the Ferrara-Porrati sum rules [18].

The vanishing of the AMM when the model has spacetime supersymmetry must be due to the contributions from the superpartners circulating in the loop cancelling one another. Therefore we expect to get zero by summing, for any given sector $m\mathbf{W}$, only over the sectors $m\mathbf{W}$ and $m\mathbf{W}+\mathbf{W}_0+\mathbf{W}_1$. We also have to implement the GSO projections by summing over the n_i. However, since the m_i and n_i are equivalent by modular invariance, we again expect to get zero only by summing over boundary conditions $n\mathbf{W}$ and $n\mathbf{W}+\mathbf{W}_0+\mathbf{W}_1$. The explicit calculations vindicate this belief. This means that it is sufficient to study the

[4] We checked this also numerically.

right-moving part of the amplitude (see subsection §1.5). It is then convenient to change basis for the $\mathbf{W}$-vectors, or equivalently, for the (m_i, n_i). We introduce then (m'_i, n'_i) by

$$
\begin{aligned}
m'_1 &= m_0 + m_1 \qquad & m'_i &= m_i \quad i \neq 1 \\
n'_1 &= n_0 + n_1 \qquad & n'_i &= n_i \quad i \neq 1 \ ,
\end{aligned}
\tag{4.5}
$$

since

$$
\begin{aligned}
m_0\mathbf{W}_0 + m_1\mathbf{W}_1 = m_0(\mathbf{W}_0 + \mathbf{W}_1) + (m_0 + m_1)\mathbf{W}_1 = \\
m'_0(\mathbf{W}_0 + \mathbf{W}_1) + m'_1\mathbf{W}_1 \ (\text{mod } 1) \ .
\end{aligned}
\tag{4.6}
$$

In the rest of this section we will work in the new basis and we will drop the primes.

The spin structures depending on (m_0, n_0) are explicitly given by

$$
\begin{aligned}
&\alpha_1 = \tfrac{1}{2}m_0 \ , \quad \alpha_2 = \tfrac{1}{2}(m_0 + m_3) \ , \quad \alpha_5 = \tfrac{1}{2}(m_0 + m_2 + m_4) \\
&\alpha_8 = \tfrac{1}{2}(m_0 + m_2 + m_3 + m_4)
\end{aligned}
\tag{4.7}
$$

and similarly for the β_i in terms of the n_i.

In a model with spacetime supersymmetry $k_{02} + k_{12} = k_{04} + k_{14}$ mod 1 and the last term in the summation coefficient (3.40) drops out so that the dependence on (m_0, n_0) is given by

$$
\begin{aligned}
K^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = \exp\big\{2\pi i\,\big[\tfrac{1}{2}(m_0 + n_0) + \tfrac{1}{2}n_0(m_2 + m_3 + m_4) \\
+ \text{ terms not depending on } (m_0, n_0)\big]\big\} \ .
\end{aligned}
\tag{4.8}
$$

Since we want to sum over (m_0, n_0) keeping fixed all the other spin structures, it is convenient to use eq. (A.3) to reexpress $\Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}$ for $l = 2, 5, 8$ in terms of $\Theta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}$. In doing so, the argument of these theta functions is shifted by $(\beta_l - \beta_1) - \tau(\alpha_l - \alpha_1)$ and we get also an overall phase $\exp\big\{2\pi i[\tfrac{1}{2}n_0(m_2 + m_3 + m_4)]\big\}$ cancelling the one appearing in (4.8), so that effectively the summation coefficients for the sum over (m_0, n_0) are reduced to

$$
\exp\big\{2\pi i\,\big[\tfrac{1}{2}(m_0 + n_0) + \text{ terms not depending on } (m_0, n_0)\big]\big\} \ . \tag{4.9}
$$

Thus we expect to prove the vanishing of the AMM in the supersymmetric case using the standard Riemann identity equation (A.10) . Notice that this is not possible in the

non-supersymmetric case since there is an extra dependence on (m_0, n_0) in the phase of the coefficient (3.40) given by

$$(k_{04} + k_{14} - k_{02} - k_{12})(m_4 n_0 + m_0 n_4) = \tfrac{1}{2}(m_4 n_0 + m_0 n_4) \text{ (mod 1)} . \tag{4.10}$$

To show that the AMM vanishes when the model is supersymmetric, it is convenient to extract all factors depending on (n_0, m_0) from eqs. (3.43), (3.45) and (3.47), arriving at the quantity

$$\begin{aligned} \sum_{m_0,n_0} K^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} & \left(\frac{1}{\log|k|} \log\left|\frac{z_1}{z_2}\right| + \frac{2}{2\pi i}\, \partial_\nu \log\Theta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} \right) \times \\ & \left(\frac{1}{\log|k|} \log\left|\frac{z_1}{z_2}\right| + \frac{2}{2\pi i}\, \partial_\nu \log\Theta\begin{bmatrix}\alpha_L\\ \beta_L\end{bmatrix}(\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} \right) \times \\ & \prod_{l=1,5} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(\tfrac{1}{2}\nu_{12}|\tau) \prod_{l=2,8} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(0|\tau) . \end{aligned} \tag{4.11}$$

Using the Riemann identity eq. (A.10) , one can prove that (in the supersymmetric case)

$$\begin{aligned} & \sum_{m_0,n_0} K^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \prod_{l=1,5} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(\tfrac{1}{2}\nu_{12}|\tau) \prod_{l=2,8} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(0|\tau) \ = \ 0 , \\ & \sum_{m_0,n_0} K^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \left(\partial_\nu \log\Theta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} + \partial_\nu \log\Theta\begin{bmatrix}\alpha_5\\ \beta_5\end{bmatrix}(\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} \right) \times \\ & \qquad \prod_{l=1,5} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(\tfrac{1}{2}\nu_{12}|\tau) \prod_{l=2,8} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(0|\tau) \ = \ 0 , \\ & \sum_{m_0,n_0} K^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}\, \partial_\nu \log\Theta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} \ \partial_\nu \log\Theta\begin{bmatrix}\alpha_5\\ \beta_5\end{bmatrix}(\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} \times \\ & \qquad \prod_{l=1,5} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(\tfrac{1}{2}\nu_{12}|\tau) \prod_{l=2,8} \Theta\begin{bmatrix}\alpha_l\\ \beta_l\end{bmatrix}(0|\tau) \ = \ 0 . \end{aligned} \tag{4.12}$$

Using these identities it is straightforward to show that the quantity (4.11) is zero, and thus the AMM vanishes due to the spacetime supersymmetry, as it follows from the sum rules [18].

In the non-supersymmetric case the quantity (4.11) is nonzero. Thus to compute the value of the AMM one also needs to sum over the other spin structures. This is particularly difficult since the sum over all the other spin structures involves more than 4 theta functions and both the right- and left-movers. Obviously it is always possible to compute numerically the AMM since the expansion in powers of $\exp[-2\pi\,\mathrm{Im}\tau]$ converges very rapidly.

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Appendix A: Notations, Conventions and Useful Formulæ

In this appendix we will give the notations and conventions we have adopted for the basic (correlation) functions on the torus. We begin by giving our conventions for the Dedekind η-function and for the theta functions.

The Dedekind η-function is given by

$$\eta(\tau) = k^{1/24} \prod_{n=1}^{\infty}(1-k^n) \ , \qquad k = e^{2\pi i\tau} \ , \tag{A.1}$$

and our conventions for the theta functions are

$$\begin{aligned}\Theta \begin{bmatrix}\alpha \\ \beta\end{bmatrix} (\nu|\tau) &= e^{i\pi(\frac{1}{2}-\alpha)^2\tau} e^{2\pi i(\frac{1}{2}+\beta)(\frac{1}{2}-\alpha)} e^{2\pi i(\frac{1}{2}-\alpha)\nu} \ \times \\ &\quad \prod_{n=1}^{\infty}(1-k^n)(1-k^{n+\alpha-1}e^{-2\pi i(\beta+\nu)})(1-k^{n-\alpha}e^{2\pi i(\beta+\nu)}) \\ &= \sum_{r\in\mathbb{Z}} e^{\pi i(r+\frac{1}{2}-\alpha)^2\tau + 2\pi i(r+\frac{1}{2}-\alpha)(\nu+\beta+\frac{1}{2})}\end{aligned} \tag{A.2}$$

$$\Theta_1 \equiv \Theta \begin{bmatrix}0 \\ 0\end{bmatrix} \ , \quad \Theta_2 \equiv \Theta \begin{bmatrix}0 \\ 1/2\end{bmatrix} \ , \quad \Theta_3 \equiv \Theta \begin{bmatrix}1/2 \\ 1/2\end{bmatrix} \ , \quad \Theta_4 \equiv \Theta \begin{bmatrix}1/2 \\ 0\end{bmatrix} \ .$$

The theta functions satisfy the following relations

$$\begin{aligned}\Theta \begin{bmatrix}\alpha+\Delta\alpha \\ \beta+\Delta\beta\end{bmatrix} (\nu|\tau) &= \exp\left[2\pi i\left\{\tfrac{1}{2}(\Delta\alpha)^2\tau - \Delta\alpha(\nu+\beta+\Delta\beta+\tfrac{1}{2})\right\}\right] \ \times \\ &\quad \Theta \begin{bmatrix}\alpha \\ \beta\end{bmatrix} (\nu - \Delta\alpha\,\tau + \Delta\beta|\tau)\end{aligned} \tag{A.3}$$

$$\Theta \begin{bmatrix}\alpha \\ \beta\end{bmatrix} (\nu|\tau) = \exp\left[2\pi i\left\{\tfrac{1}{2}\alpha^2\tau - \alpha(\nu+\beta+\tfrac{1}{2})\right\}\right] \Theta_1(\nu-\alpha\tau+\beta|\tau) \tag{A.4}$$

and

$$\Theta \begin{bmatrix}\alpha+m \\ \beta+n\end{bmatrix} (\nu|\tau) = \exp\left[2\pi i(\tfrac{1}{2}-\alpha)n\right] \Theta \begin{bmatrix}\alpha \\ \beta\end{bmatrix} (\nu|\tau) \tag{A.5}$$

$$\begin{aligned}&\Theta \begin{bmatrix}\alpha \\ \beta\end{bmatrix} (\nu+m\tau+n|\tau) = \\ &\quad \exp\left[2\pi i\left\{-\tfrac{1}{2}m^2\tau + (\tfrac{1}{2}-\alpha)n - m\nu - m(\beta+\tfrac{1}{2})\right\}\right] \Theta \begin{bmatrix}\alpha \\ \beta\end{bmatrix} (\nu|\tau) \ ,\end{aligned} \tag{A.6}$$

where m, n are integer numbers.

The bosonic Green function on the torus is given by

$$G_B(z_1, \bar{z}_1; z_2, \bar{z}_2) = 2\left[\log|E(z_1, z_2)| - \tfrac{1}{2}\mathrm{Re}\left(\int_{z_2}^{z_1}\omega\right)^2 \frac{1}{2\pi \mathrm{Im}\tau}\right] \tag{A.7}$$

and the prime form is

$$E(z_1, z_2) = \frac{2\pi i \Theta_1(\nu_{12}|\tau)}{\sqrt{\omega(z_1)\omega(z_2)}\Theta_1'(0|\tau)} \ , \qquad \nu_{12} = \int_{z_2}^{z_1} \frac{\omega}{2\pi i} \ , \tag{A.8}$$

where $\omega(z)$ is the holomorphic 1-form on the torus, normalized to have period $2\pi i$ around the a-cycle. In the parametrization where $\omega(z) = 1/z$ the prime form (A.8) becomes

$$E(z_1, z_2) = (z_1 - z_2)\prod_{n=1}^{\infty} \frac{(1 - \frac{z_1}{z_2}k^n)(1 - \frac{z_2}{z_1}k^n)}{(1 - k^n)^2} \ . \tag{A.9}$$

The standard Riemann identity is ($\alpha, \beta = \{0, \frac{1}{2}\}$)

$$\sum_{\alpha,\beta} e^{2\pi i(\alpha+\beta)} \prod_{i=1}^{4} \Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](x_i|\tau) \ = \ 0 \ , \tag{A.10}$$

where one of the following equations must hold

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 = 0 \qquad && x_1 - x_2 - x_3 + x_4 = 0 \\ &x_1 - x_2 + x_3 - x_4 = 0 \qquad && x_1 + x_2 - x_3 - x_4 = 0 \ . \end{aligned} \tag{A.11}$$

Having given our notations for the basic functions on the torus, we can now turn to the correlators.

The spacetime coordinate fields X^μ satisfy the OPE

$$X^\mu(z, \bar{z})X^\nu(w, \bar{w}) \overset{OPE}{=} -\delta^{\mu\nu}\left(\log(z - w) + \log(\bar{z} - \bar{w})\right) + \ \cdots \ . \tag{A.12}$$

Their one-loop partition function is given by

$$Z_X = \prod_{n=1}^{\infty} |1 - k^n|^{-8}(2\pi \mathrm{Im}\tau)^{-2} \ , \tag{A.13}$$

and the genus one correlator is

$$\langle\langle\, X^\mu(z, \bar{z})X^\nu(w, \bar{w})\,\rangle\rangle \ = \ -\delta^{\mu\nu}G_B(z, \bar{z}; w, \bar{w}) \ Z_X \ . \tag{A.14}$$

(notice that this correlator does not decompose in the product of a holomorphic times an anti-holomorphic part, only $\partial_w \partial_z \langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w})\rangle$ is holomorphic).

For the world-sheet fermions we have the following normalization

$$\psi^\mu(z)\psi^\nu(w) \stackrel{OPE}{=} \frac{\delta^{\mu\nu}}{z-w} + \cdots \tag{A.15}$$
$$\psi^m_{(l)}(z)\psi^n_{(k)}(w) \stackrel{OPE}{=} \frac{\delta^{m,n}\delta_{l,k}}{z-w} + \cdots .$$

They are bosonized according to eq. (2.4) and correlation functions are defined as in eq. (3.10). The fundamental genus one correlator [25] is

$$\langle \prod_{i=1}^{N} e^{q_i \phi(z_i)} \rangle \left[{\alpha \atop \beta}\right] = \delta_{\sum_{i=1}^N q_i, 0} \prod_{i<j} [E(z_i,z_j)]^{q_i q_j} \; \Theta\left[{\alpha \atop \beta}\right] \left(\sum_{i=1}^{N} q_i \int^{z_i} \tfrac{\omega}{2\pi i} |\tau \right) , \tag{A.16}$$

where we have explicitly displayed the spin structure dependence of the correlator, whereas in the paper we often adopt the following short-hand notation

$$\langle \prod_{i=1}^{N} e^{q_i \phi(z_i)} \rangle_{(l)} = \langle \prod_{i=1}^{N} e^{q_i \phi(z_i)} \rangle \left[{\alpha_l \atop \beta_l}\right] . \tag{A.17}$$

We use also the following notation

$$\langle S_+(z_1) S_-(z_2) \rangle = \langle e^{\frac{1}{2}\phi(z_1)} e^{-\frac{1}{2}\phi(z_2)} \rangle = (E(z_1,z_2))^{-\frac{1}{4}} \, \Theta\left[{\alpha \atop \beta}\right] \left(\tfrac{1}{2}\nu_{12}|\tau\right) , \tag{A.18}$$

and we define the integer

$$S_l \equiv (1-2\alpha_l)(1+2\beta_l) , \tag{A.19}$$

which is even (odd) whenever the (l)th spin structure is even (odd).

Appendix B: Ghost and Superghost Correlators

Since we choose to work in the Lorentz-covariant formulation, all 1-loop computations involve also the calculation of certain ghost and superghost correlators on the torus.

The ghost correlator relevant for our 1-loop scattering amplitudes involving N physical external states is

$$\mathrm{d}^2 k \prod_{i=1}^{N-1} \mathrm{d}^2 z_i \; \langle\langle \left| (\eta_k|b) \prod_{i=1}^{N-1} (\eta_{z_i}|b) \prod_{i=1}^{N} c(z_i) \right|^2 \rangle\rangle \tag{B.1}$$
$$= \frac{\mathrm{d}^2 k}{\bar{k}^2 k^2} \prod_{i=1}^{N-1} \mathrm{d}^2 z_i \left| \frac{1}{\omega(z_N)} \right|^2 \prod_{n=1}^{\infty} |1-k^n|^4 .$$

Here η_k, η_{z_i} are the Beltrami-differentials dual to the moduli k $(= e^{2\pi i\tau})$ and z_i, and the point z_N has been fixed using the translational invariance of the torus. ω is the holomorphic one-form, normalized to have period $2\pi i$ around the a-cycle. If the Picture Changing Operators are inserted at arbitrary points on the torus other ghost correlators besides (B.1) will in general be needed [23]. However, for the 3-point calculations we consider in this paper the correlator (B.1) will suffice.

The superghost correlators are most conveniently calculated in the "bosonized" formalism [14]. They have been given up to overall numerical factors in ref. [26,27]. However, such overall factors are not unimportant since they may in general depend on the spin structure. Therefore, we compute all superghost correlators using the N-point g-loop vertex for the bosonized (β,γ)-system [19] which has been obtained by the sewing technique [28] and therefore automatically includes all phase factors required by factorization.

As an example, consider the correlation function

$$\begin{aligned}\langle\langle \prod_{i=1}^{N} e^{q_i\phi(z_i)}\, \xi(z_{N+1})\,\rangle\rangle \; = & \\ & \prod_{n=1}^{\infty}(1-k^n)\prod_{i=1}^{N}(\sigma(z_i))^{-2q_i}\prod_{i<j}(E(z_i,z_j))^{-q_iq_j}\;\times \\ & \left[\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\left(-\sum_{j=1}^{N} q_j\int_{z_0}^{z_j}\frac{\omega}{2\pi i} + 2\Delta^{z_0}|\tau\right)\right]^{-1} ,\end{aligned} \tag{B.2}$$

where $q_1 + \ldots + q_N = 0$ and the insertion of the operator $\xi(z_{N+1})$ is needed to saturate the integration over the ξ zero mode, a degree of freedom not present in the original (β,γ) system. The result (B.2) is obtained by saturating the vertex given by eq. (6.10) of ref. [19] with the following $N+1$ highest weight states [5]

$$e^{q_i\phi^{(i)}(0)}|0\rangle_i \quad \text{for} \quad i=1,\ldots,N \qquad\qquad \text{and} \qquad\qquad e^{\chi^{(N+1)}(0)}|0\rangle_{N+1} \; . \tag{B.3}$$

The result (B.2) agrees with eq. (36) of ref. [26] *except* for an overall minus sign whenever the spin structure is odd. Since the parity of the spin structure is modular invariant such a sign can never be fixed by modular invariance, only by factorization.

At genus one

$$\Delta^{z_0} = -\frac{1}{4\pi i}\log k = -\frac{1}{2}\tau \; , \tag{B.4}$$

[5] Assuming $V_i'(0) = 1$ for simplicity.

and σ is a multivalued 1/2-differential which reduces to

$$\sigma(z) = 1 \tag{B.5}$$

if we choose coordinates such that $\omega(z) = 1/z$.

Using eq. (A.6) for the Θ-functions we arrive at our final expression for the correlator (B.2)

$$\begin{aligned} \langle\langle \prod_{i=1}^{N} e^{q_i\phi(z_i)}\ \xi(z_{N+1})\,\rangle\rangle \ = \qquad\qquad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathrm{B.6}) \\ (-1)^S k^{1/2} \prod_{n=1}^{\infty}(1-k^n) \prod_{i=1}^{N}(\omega(z_i))^{-q_i} \prod_{i<j}(E(z_i,z_j))^{-q_iq_j} \ \times & \\ \left[\Theta\left[{}^{\alpha}_{\beta}\right]\left(\sum_{j=1}^{N} q_j \int_{z_0}^{z_j} \frac{\omega}{2\pi i}|\tau\right)\right]^{-1} \quad . & \end{aligned}$$

Here we dropped a phase factor $\exp\{2\pi i(1/2+\beta)\}$ which is already included in the KLT summation coefficient (1.17). The remaining phase factor $(-1)^S$, where $S = (1-2\alpha)(1+2\beta)$ is even (odd) whenever the spin structure is even (odd), is crucial in order to obtain a vanishing Anomalous Magnetic Moment in the supersymmetric case, as discussed in section §4.

Appendix C: Covariant Formulation of KLT Formalism

In this appendix we outline the modifications encountered when rephrasing the Kawai-Lewellen-Tye 4-dimensional string models in a Lorentz-covariant way. In particular, we obtain the covariant form of the GSO projections.

The Kawai-Lewellen-Tye [11] construction of 4-dimensional string theories is performed in the light-cone gauge. We have 22 left-moving and 10 right-moving complex fermions, whose 1-loop partition function is given by

$$Z_{\text{fermion}} = \sum_{n_i,m_j} \tilde{C}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} Z^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \ , \tag{C.1}$$

where

$$Z^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = \mathrm{Tr}\left[\left(\prod_{l=1}^{22} \bar{k}^{\overline{H}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]}}\right)\left(\prod_{l=1}^{10} k^{H^{(l)}_{[\![\alpha_l]\!]}}\right) e^{2\pi i(\mathbf{W}_0+\boldsymbol{\beta})\cdot\mathbf{N}_{[\![\boldsymbol{\alpha}]\!]}}\right] \tag{C.2}$$

is the partition function corresponding to a given set of spin structures, specified as in eq. (1.15) by the integers m_i and n_i. The summation coefficients

$$\tilde{C}^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = \frac{1}{\prod_i M_i} \exp\{-2\pi i[\sum_i (n_i + \delta_{i,0})(\sum_j k_{ij} m_j + s_i + k_{0i} - \mathbf{W}_i \cdot [\![\boldsymbol{\alpha}]\!]) + \sum_i m_i s_i + \tfrac{1}{2}]\} \quad \text{(C.3)}$$

are carefully constructed to ensure modular invariance of Z_{fermion}, as well as the correct spin-statistics relation (i.e. space-time bosons (fermions) contribute with weight $+1$ (-1) to the partition function). $N_{[\![\boldsymbol{\alpha}]\!]}$ is the vector of fermion number operators, given by eq. (1.11) and the Hamiltonians are given by

$$H^{(l)}_{[\![\alpha_l]\!]} = \tilde{H}^{(l)}_{[\![\alpha_l]\!]} + \tfrac{1}{2}((\alpha_l)^2 - \alpha_l + \frac{1}{6}) \ , \quad \text{(C.4)}$$

where

$$\tilde{H}^{(l)}_{[\![\alpha_l]\!]} = \sum_{q=1}^{\infty} ((q + [\![\alpha_l]\!] - 1) n^{(l)}_{q+[\![\alpha_l]\!]-1} + (q - [\![\alpha_l]\!]) n^{(l)*}_{q-[\![\alpha_l]\!]}) \ . \quad \text{(C.5)}$$

We may now add the longitudinal complex fermion, $\psi_{(0)}$, and the superghosts β and γ. By world-sheet supersymmetry, both carry the same spin structure as the transverse complex fermion, $\psi_{(1)}$. The corresponding partition functions are

$$Z_{\psi_{(0)}} = \mathrm{Tr}\left[k^{\tilde{H}^{(0)}_{[\![\alpha_1]\!]}} e^{-2\pi i(1/2+\beta_1) N^{(0)}_{[\![\alpha_1]\!]}}\right] \quad \text{(C.6)}$$

$$Z_{(\beta\gamma)} = \mathrm{Tr}\left[k^{\tilde{H}^{(\beta\gamma)}_{[\![\alpha_1]\!]}} e^{2\pi i \beta_1 N^{(\beta\gamma)}_{[\![\alpha_1]\!]}}\right] \ , \quad \text{(C.7)}$$

where $N^{(\beta\gamma)}_{[\![\alpha_1]\!]}$ is given by eq. (1.13) and

$$\tilde{H}^{(\beta\gamma)}_{[\![\alpha_1]\!]} = \sum_{q=1}^{\infty} \left[(q - 1 + [\![\alpha_1]\!]) \beta_{-q+1-[\![\alpha_1]\!]} \gamma_{q-1+[\![\alpha_1]\!]} - (q - [\![\alpha_1]\!]) \gamma_{-q+[\![\alpha_1]\!]} \beta_{q-[\![\alpha_1]\!]}\right] \quad \text{(C.8)}$$

corresponding to the choice of superghost vacuum $|q' = -1/2 - [\![\alpha_1]\!]\rangle$, i.e. the superghost part of the ground state vertex operator is $e^{-\phi}$ in a bosonic sector and $e^{-\phi/2}$ in a fermionic sector.

We have carefully chosen the definitions (C.6) and (C.7) to ensure that

$$Z_{\psi_{(0)}} Z_{(\beta\gamma)} = 1 \quad \text{(C.9)}$$

— if the two factors did not cancel each other completely, their inclusion would alter the already correct result (C.1).

Using the expression (C.3) for the summation coefficients, the partition function (C.1) can now be written as

$$Z_{\text{fermion}} = \frac{1}{\prod_i M_i} \sum_{m_i,n_i} e^{-2\pi i[\sum_i (n_i+\delta_{i,0})(\sum_j k_{ij}m_j+s_i+k_{0i}-\mathbf{W}_i\cdot[\![\boldsymbol{\alpha}]\!])+\sum_i m_i s_i+1/2]}$$
$$\mathrm{Tr}\left[\left(\prod_{l=1}^{22} \bar{k}^{\overline{H}^{(\bar{l})}_{[\![\bar{\alpha}_l]\!]}}\right)\left(\prod_{l=1}^{10} k^{H^{(l)}_{[\![\alpha_l]\!]}}\right) k^{\tilde{H}^{(0)}_{[\![\alpha_1]\!]}} k^{\tilde{H}^{(\beta\gamma)}_{[\![\alpha_1]\!]}} e^{2\pi i\sum_i (n_i+\delta_{i,0})\mathbf{W}_i\cdot\mathbf{N}_{[\![\boldsymbol{\alpha}]\!]}}\right.$$
$$\left. e^{-2\pi i(1/2+\sum_i n_i s_i)N^{(0)}_{[\![\alpha_1]\!]}} e^{2\pi i\sum_i n_i s_i N^{(\beta\gamma)}_{[\![\alpha_1]\!]}}\right] . \tag{C.10}$$

Summing over the n_i enforces the GSO projections in the loop. We therefore arrive at the covariant form of the GSO projection conditions

$$\mathbf{W}_i \cdot \mathbf{N}_{[\![\boldsymbol{\alpha}]\!]} - s_i(N^{(0)}_{[\![\alpha_1]\!]} - N^{(\beta\gamma)}_{[\![\alpha_1]\!]}) \overset{\text{MOD } 1}{=} \sum_j k_{ij}m_j + s_i + k_{0i} - \mathbf{W}_i \cdot [\![\boldsymbol{\alpha}]\!] . \tag{C.11}$$

The physical external states are in the superghost vacuum (with charge -1 or $-1/2$) and therefore have $N^{(\beta\gamma)}_{[\![\alpha_1]\!]} = 0$.

Appendix D: Normalization of the "Electron/Positron" Vertex Operator

Factorization dictates that the problem of normalizing string amplitudes can be separated into two independent problems: One, to fix the normalization constant C_g of the vacuum amplitude at genus g. The other, to fix the normalization of each vertex operator in the theory.

The constant C_g as well as the overall normalization of the photon vertex operator has been derived in ref. [17] and are given by eqs. (2.3) (for $g = 1$) and (2.56) respectively. In this appendix we consider the normalization of the fermion vertex operator (2.58)

$$\mathcal{V}^{-1/2}(z,\bar{z};k;\mathbb{V}) = N_f \overline{\mathbf{V}}^{\bar{a}} \bar{S}_{\bar{a}}(\bar{z}) \mathbf{V}^a S_a(z) e^{-\phi(z)/2}(c_{(11)})^{-1/2} e^{ik\cdot X(z,\bar{z})} . \tag{D.1}$$

Obviously, the quantity N_f is not well defined until we have specified our conventions for the spinors $\overline{\mathbf{V}}^{\bar{a}}$ and $\mathbf{V}^a$. This we will do in the following way: First we define the spinors in the case of any incoming string state (D.1) with negative $U(1)$ charge (any incoming "electron"). Next, we define the anti-particle state ("positron") corresponding to each of

these incoming "electron" states. Finally we define the out-going state corresponding to any incoming state.

Having carefully normalized the spinors we may determine N_f using the method of ref. [17]: We consider the elastic scattering of a photon and an "electron" at very high center-of-mass energies, where the interactions are dominated by gravity, and require that the tree-level amplitude for this process should reproduce the standard one dictated by the principle of equivalence. This will yield an expression for N_f in terms of the gravitational coupling κ. Finally, to relate κ to the $U(1)$ charge, e, we consider the tree-level amplitude of two "electrons" and a photon and require that we reproduce the standard Yukawa coupling.

We write the spinor pertaining to an incoming "electron" as follows:

$$\mathbb{V}_{\rm in}^{-}(k,s;\{\bar{q}_l\},f) \equiv \overline{\mathbf{V}}^{-}(\{\bar{q}_l\}) \otimes \mathbf{P} V_{\rm in}^{-}(k,s) \otimes v_{(4)}^{-} \otimes v_{(567)}^{-}(\{\bar{q}_l\},f) \otimes v_{(9)}^{f} \ . \tag{D.2}$$

Here we imagine $\overline{\mathbf{V}}^{-}(\{\bar{q}_l\})$ to be an eigenvector of the various "charge" operators $\sigma_3^{(l)}$, with eigenvalues $\bar{q}_l$, $l = 1,\dots,7;15,16$ and with $U(1)$-charge $\sigma_3^{(17)} = -1$, normalized such that

$$\Big(\overline{\mathbf{V}}^{-}(\{\bar{q}_l\})\Big)^{*} = \overline{\mathbf{V}}^{-}(\{\bar{q}_l\}) \qquad \text{and} \qquad \Big(\overline{\mathbf{V}}^{-}(\{\bar{q}_l\})\Big)^{\dagger}\overline{\mathbf{V}}^{-}(\{\bar{q}_l\}) = 1 \ . \tag{D.3}$$

The projection operator

$$\mathbf{P} = \tfrac{1}{2}(\mathbf{1} - \exp\{2\pi i[k_{00}+k_{01}+k_{03}+k_{13}]\}\Gamma^5\sigma_3^{(5)}) \tag{D.4}$$

enforces the first of the four GSO conditions given by eqs. (1.41). The space-time spinor $V_{\rm in}^{-}(k,s)$ satisfies the equation

$$(\not{k}^T - i\,(\text{sign})\frac{1}{\sqrt{2}})V_{\rm in}^{-}(k,s) = 0 \tag{D.5}$$

with [6]

$$(\text{sign}) = -\exp\{2\pi i[k_{00}+k_{01}+k_{02}+k_{03}+k_{04}+k_{12}+k_{14}+k_{23}+k_{34}]\} \ . \tag{D.6}$$

If we define new gamma matrices by

$$\gamma^{\mu} \equiv -i(\Gamma^{\mu})^T \ , \tag{D.7}$$

[6] In this appendix we adopt the cocycle choice (2.47) throughout.

eq. (D.5) becomes the ordinary Dirac equation (in the notation of ref. [29]) and we may identify $V^-_{\rm in}(k,s)$ with the standard u spinors as follows

$$V^-_{\rm in}(k,s) = \begin{cases} u(p,s) & \text{if} \quad (\text{sign}) = +1 \\ \Gamma^5 u(p,s) & \text{if} \quad (\text{sign}) = -1 \end{cases} , \tag{D.8}$$

where $p = \sqrt{\frac{\alpha'}{2}}k$ is the dimensionful momentum and $s = \pm 1/2$ is the spin in the rest frame.

The second GSO condition in (1.41) specifies $\sigma_3^{(4)}$ in terms of $\sigma_3^{(\overline{17})} = -1$, and hence $v^-_{(4)}$ up to an overall constant. Likewise, the "family label" $f = \pm 1$, defined as the eigenvalue of $\sigma_3^{(9)}$, specifies $v^f_{(9)}$ up to a normalization constant.

We may normalize $v^-_{(4)}$ and $v^f_{(9)}$ in analogy with (D.3)

$$(v^-_{(4)})^* = v^-_{(4)} \qquad \text{and} \qquad (v^-_{(4)})^\dagger v^-_{(4)} = 1 \tag{D.9}$$

$$(v^f_{(9)})^* = v^f_{(9)} \qquad \text{and} \qquad (v^f_{(9)})^\dagger v^f_{(9)} = 1 \ . \tag{D.10}$$

Finally, the spinor in the spaces (5), (6) and (7), $v^-_{(567)}(\{\bar{q}_l\}; f)$, satisfies the "mass eigenvalue equation"

$$-\tfrac{1}{2}(\sigma_2^{(5)}\sigma_1^{(6)}\sigma_1^{(7)} + \sigma_1^{(5)}\sigma_2^{(6)}\sigma_2^{(7)})\ v^-_{(567)}(\{\bar{q}_l\}; f) = \frac{1}{\sqrt{2}}\, v^-_{(567)}(\{\bar{q}_l\}; f) \tag{D.11}$$

and is in fact specified up to an overall constant by the third and fourth GSO projections in (1.41), once all charges and the "family label" f have been specified. It would be inconsistent with eq. (D.11) to take $v^-_{(567)}(\{\bar{q}_l\}; f)$ to be real. Instead it is consistent to impose the Majorana-like condition

$$\sigma_3^{(5)} C_{(567)} v^-_{(567)} = (v^-_{(567)})^* \tag{D.12}$$

and

$$(v^-_{(567)})^\dagger v^-_{(567)} = 1 \ . \tag{D.13}$$

Here $C_{(567)} \equiv \sigma_1^{(5)}\sigma_2^{(6)}\sigma_1^{(7)}$. Notice that by the antisymmetry of $C_{(567)}$

$$(v^-_{(567)})^T C_{(567)} v^-_{(567)} = 0 \ . \tag{D.14}$$

Given the spinor (D.2), describing an incoming "electron" string state with $SO(14)$ and $SO(4)$ charges $\{\bar{q}_l\}$, family label f, momentum p and spin s, we define the corresponding incoming "positron" string state by

$$\mathbb{V}^+_{\rm in}(k,s;\{-\bar{q}_l\}; -f) = \boldsymbol{\Sigma}\, \mathbb{V}^-_{\rm in}(k,s;\{\bar{q}_l\}; f) \ , \tag{D.15}$$

where the operator

$$\boldsymbol{\Sigma} \equiv (\text{sign})\ C_{SO(14)}\Gamma_{SO(14)} \otimes C_{SO(4)}\Gamma_{SO(4)} \otimes \sigma_1^{(17)}\sigma_3^{(17)} \otimes \sigma_1^{(4)} \otimes \sigma_3^{(5)} \otimes \sigma_1^{(9)} \tag{D.16}$$

changes sign on all charges, as well as the "family label", and commutes with the GSO projection operators.

Similarly, given any incoming "electron" or "positron" state with certain $SO(14)$ and $SO(4)$ charges $\{\bar{q}_l\}$, "family label" f, four-momentum p and spin s, we define the outgoing state with the same quantum numbers by

$$\mathbb{V}^{\pm}_{\text{out}}(-k, s; \{\bar{q}_l\}, f) = \mathbb{T}\left(\mathbb{V}^{\pm}_{\text{in}}(k, s; \{\bar{q}_l\}; f)\right)^* \ , \tag{D.17}$$

where

$$\mathbb{T} \equiv -e^{-i\pi\varphi_c}\ \Gamma_{SO(14)}\,\Gamma_{SO(4)}\,\sigma_3^{(17)}\,\mathbb{C}\,\Gamma^0 \tag{D.18}$$

also commutes with all GSO projection operators. The definitions (D.16) and (D.18) are chosen so as to reproduce standard field theory results for the pair annihilation and brehmsstrahlung tree-level amplitudes, see below.

We may now compute the tree-level amplitudes of various processes, using the formula (in the notation of section §2):

$$T^{\text{tree}}(\lambda_1, \ldots, \lambda_{N_{\text{out}}}|\lambda_{N_{\text{out}}+1}, \ldots, \lambda_{N_{\text{out}}+N_{\text{in}}}) \ = \ -\frac{4\pi^3}{\alpha'\kappa^2}\int \prod_{i=1}^{N_{\text{tot}}-3} \mathrm{d}^2 z_i \tag{D.19}$$

$$\langle\langle \left| \prod_{i=1}^{N_{\text{tot}}-3} (\eta_{z_i}|b) \prod_{i=1}^{N_{\text{tot}}} c(z_i) \right|^2 \prod_{A=1}^{N_B+N_{FP}-2} \Pi(w_A)\ \mathcal{V}_{\langle\lambda_1|}(z_1,\bar{z}_1)\ldots\mathcal{V}_{|\lambda_{N_{\text{tot}}}\rangle}(z_{N_{\text{tot}}},\bar{z}_{N_{\text{tot}}})\rangle\rangle \ .$$

For the elastic scattering of "electrons/positrons" with photons we find in the limit $s \to \infty$, $t \to 0$ (where s and t are the usual Mandelstam variables) the correct result [30]

$$\langle\gamma\ e^{\pm}|T|\gamma\ e^{\pm}\rangle_{\text{tree}} = -\kappa^2\frac{s^2}{t} \tag{D.20}$$

if we make the identification

$$(N_f)^2 = -\frac{\kappa^2\sqrt{\alpha'}}{\pi^2}e^{-i\pi\varphi_c} \ . \tag{D.21}$$

For the pair-annihilation and brehmsstrahlung processes we recover the standard field theory results

$$T^{\text{tree}}(\gamma|e^{\pm}\ e^{\mp}) = \pm e u^T(p_1, s_1)C\gamma^{\mu}\epsilon_{\mu}u(p_2, s_2) \tag{D.22}$$

$$T^{\text{tree}}(e^{\pm}\ \gamma|e^{\pm}) = \pm e u^{\dagger}(-p_1, s_1)\gamma^0\gamma^{\mu}\epsilon_{\mu}u(p_2, s_2) \ , \tag{D.23}$$

where $e = -|e|$ is the $U(1)$ charge of the "electron", if we identify

$$|e| = \frac{\kappa}{\sqrt{2\alpha'}} \ . \tag{D.24}$$

Appendix E: Proof of the equality of the two expressions for the $I\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ function

In this appendix we present a proof of the simplest of the identities in theta functions that we encountered in subsection §3.3, arising from the requirement of Lorentz covariance of the genus one correlators appearing in the one-loop three-point amplitude. The other identities needed in that computation can be proven in a similar way.

The identity we want to prove can be stated as the fact that on the torus

$$L\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2|\tau) \;=\; R\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2|\tau)\ , \tag{E.1}$$

where

$$\begin{aligned}
L\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2|\tau) &= \partial_z \log\frac{E(z,z_1)}{E(z,z_2)} + 2\frac{\omega(z)}{2\pi i}\partial_\nu \log\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} \\
R\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2|\tau) &= \frac{E(z_1,z_2)}{E(z,z_1)E(z,z_2)}\left(\frac{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\nu_z-\frac{1}{2}\nu_{z_1}-\frac{1}{2}\nu_{z_2}|\tau)}{\Theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\frac{1}{2}\nu_{12}|\tau)}\right)^2 \\
\nu_z &= \int^z \frac{\omega}{2\pi i} \\
\nu_{12} &= \int_{z_2}^{z_1}\frac{\omega}{2\pi i}\ .
\end{aligned} \tag{E.2}$$

The proof consists in showing that L and R, considered as meromorphic one-forms in z (for every fixed value of z_1, z_2 and τ), have the same periodicity properties, zeros, poles and residues at the poles. Then

$$\frac{L\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2|\tau)}{R\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z,z_1,z_2|\tau)} \tag{E.3}$$

is a single-valued, globally holomorphic function of z, that is a constant. Since L and R have the same residues, the constant is 1, thus proving the identity. So, we need to study the periodicity properties, zeros and poles of R and L as one-forms in z.

First of all, using the formulæ given in Appendix A, we can rewrite L and R as follows

$$L \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (z, z_1, z_2|\tau) = \frac{\omega(z)}{2\pi i} \Big(\partial_\nu \log \Theta_1(\nu|\tau)|_{\nu=\nu_z-\nu_{z_1}} - \\ \partial_\nu \log \Theta_1(\nu|\tau)|_{\nu_z-\nu_{z_2}} + 2\partial_\nu \log \Theta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\nu|\tau)\Big|_{\nu=\frac{1}{2}\nu_{12}} \Big) \tag{E.4}$$

$$R \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (z, z_1, z_2|\tau) = \frac{\omega(z)}{2\pi i} \frac{\Theta_1(\nu_{z_1}-\nu_{z_2}|\tau)\Theta_1'(0|\tau)}{\Theta_1(\nu_z-\nu_{z_1}|\tau)\Theta_1(\nu_z-\nu_{z_2}|\tau)} \times \\ \left(\frac{\Theta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\nu_z - \frac{1}{2}\nu_{z_1} - \frac{1}{2}\nu_{z_2}|\tau)}{\Theta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\frac{1}{2}\nu_{12}|\tau)} \right)^2 .$$

Now, using formula (A.6) one can show that L and R are both single-valued on the torus, that is, under the shift $\nu_z \to \nu_z + m\tau + n$ (m, n integers) they are invariant.

As a function of z, it is obvious from eq. (E.2) that both L and R have poles at $z = z_1$ and $z = z_2$, with residues $+1$ and -1 respectively.

Thus, what is left to be proven is that L and R have the same zeros. For R, a zero in z can come only from the factor

$$\left(\Theta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\nu_z - \tfrac{1}{2}\nu_{z_1} - \tfrac{1}{2}\nu_{z_2}|\tau)\right)^2 . \tag{E.5}$$

Using formula (A.4) and the fact that $\Theta_1(m\tau + n|\tau) = 0$, we get that R has a double zero when

$$\nu_z - \tfrac{1}{2}\nu_{z_1} - \tfrac{1}{2}\nu_{z_2} - \alpha\tau + \beta + m\tau + n \ = \ 0 \tag{E.6}$$

or

$$\nu_z \ = \ \tfrac{1}{2}\nu_{z_1} + \tfrac{1}{2}\nu_{z_2} + \alpha\tau - \beta - m\tau - n \ \equiv \ \nu_0 - m\tau - n \ . \tag{E.7}$$

We now look for the zeros of L. First it is convenient to write

$$\partial_\nu \log \Theta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (\nu|\tau) \ = \ \partial_\nu \log \Theta_1(\nu - \alpha\tau + \beta|\tau) - 2\pi i\alpha \tag{E.8}$$

and to introduce

$$x \ = \ \nu_z - \nu_0 \ . \tag{E.9}$$

Thus

$$L \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] (z, z_1, z_2|\tau) = \frac{\omega(z)}{2\pi i} \Big(\partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=x-\frac{1}{2}\nu_{12}+\alpha\tau-\beta} - \\ \partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=x+\frac{1}{2}\nu_{12}+\alpha\tau-\beta} + 2\partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=\frac{1}{2}\nu_{12}-\alpha\tau+\beta} - 4\pi i\alpha \Big) \ . \tag{E.10}$$

Now, as long as 2α and 2β are integers, it follows from (A.6) that

$$\begin{aligned}\partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=x+\frac{1}{2}\nu_{12}+\alpha\tau-\beta} &= \partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=x+\frac{1}{2}\nu_{12}-\alpha\tau+\beta} - 4\pi i\alpha \\ \partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=x-\frac{1}{2}\nu_{12}+\alpha\tau-\beta} &= -\partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=-x+\frac{1}{2}\nu_{12}-\alpha\tau+\beta}\end{aligned} \tag{E.11}$$

from which we get

$$\begin{aligned}L\left[{\alpha \atop \beta}\right](z, z_1, z_2|\tau) = \frac{\omega(z)}{2\pi i}\Bigg(&-\partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=-x+\frac{1}{2}\nu_{12}-\alpha\tau+\beta} - \\ &\partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=x+\frac{1}{2}\nu_{12}-\alpha\tau+\beta} + 2\partial_\nu \log \Theta_1(\nu|\tau)\big|_{\nu=\frac{1}{2}\nu_{12}-\alpha\tau+\beta}\Bigg) .\end{aligned} \tag{E.12}$$

Thus L vanishes when $x = 0$, and then, by periodicity on the torus, whenever equation (E.6) holds. Since L is a meromorphic one-form on the torus, the number of zeros (counted with multiplicities) must be equal to the number of poles, that is 2. Now, the quantity in the bracket in eq. (E.12) is an even function of x, so the zero $x = 0$ must be at least of second order. This shows that the zero is precisely second order and also, that there are no other zeros.

Thus, we have proven that the one-forms $L(z)$ and $R(z)$ have the same zeros, poles and residues at the poles. This concludes our proof of the identity (E.3).

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